ADP-12-09/T776

$\Delta I = 1/2$ rule for kaon decays derived from QCD infrared fixed point

R. J. Crewther[1,*] and Lewis C. Tunstall[1,2,3,†]

[1] CSSM and ARC Centre of Excellence for Particle Physics at the Tera-scale, Department of Physics, University of Adelaide, Adelaide, South Australia 5005, Australia
[2] Berkeley Center for Theoretical Physics, Department of Physics, University of California, Berkeley, California 94720, USA
[3] Albert Einstein Centre for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland

This article gives details of our proposal to replace ordinary chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT$_3$ by three-flavor chiral-scale perturbation theory χPT$_\sigma$. In χPT$_\sigma$, amplitudes are expanded at low energies and small u, d, s quark masses about an infrared fixed point $\alpha_{\rm IR}$ of three-flavor QCD. At $\alpha_{\rm IR}$, the quark condensate $\langle \bar{q}q \rangle_{\rm vac} \neq 0$ induces *nine* Nambu-Goldstone bosons: π, K, η and a 0^{++} QCD dilaton σ. Physically, σ appears as the $f_0(500)$ resonance, a pole at a complex mass with real part $\lesssim m_K$. The $\Delta I = 1/2$ rule for nonleptonic K-decays is then a *consequence* of χPT$_\sigma$, with a $K_S\sigma$ coupling fixed by data for $\gamma\gamma \to \pi\pi$ and $K_S \to \gamma\gamma$. We estimate $R_{\rm IR} \approx 5$ for the nonperturbative Drell-Yan ratio $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ at $\alpha_{\rm IR}$ and show that, in the many-color limit, σ/f_0 becomes a narrow $q\bar{q}$ state with planar-gluon corrections. Rules for the order of terms in χPT$_\sigma$ loop expansions are derived in Appendix A and extended in Appendix B to include inverse-power Li-Pagels singularities due to external operators. This relates to an observation that, for $\gamma\gamma$ channels, partial conservation of the dilatation current is *not* equivalent to σ-pole dominance.

PACS numbers: 12.38.Aw, 13.25.Es, 11.30.Na, 12.39.Fe
Keywords: Nonperturbative QCD, Infrared fixed point, Dilaton, Chiral lagrangians, Nonleptonic kaon decays

I. SUMMARY

The precise determination of the mass and width of the $f_0(500)$ resonance [1–3] prompted us [4] to revisit an old idea [5, 6] that the chiral condensate $\langle \bar{q}q \rangle_{\rm vac} \neq 0$ may also be a condensate for scale transformations in the chiral $SU(3)_L \times SU(3)_R$ limit. This may occur in QCD if the heavy quarks t, b, c are first decoupled and then the strong coupling[1] α_s of the resulting three-flavor theory runs nonperturbatively to a fixed point $\alpha_{\rm IR}$ in the infrared limit (Fig. 1). At that point, $\beta(\alpha_{\rm IR})$ vanishes, so the gluonic term in the strong trace anomaly [7]

$$\theta^\mu_\mu = \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^{a\mu\nu} + \bigl(1 + \gamma_m(\alpha_s)\bigr) \sum_{q=u,d,s} m_q \bar{q}q \quad (1)$$

is absent, which implies

$$\begin{aligned} \theta^\mu_\mu\big|_{\alpha_s = \alpha_{\rm IR}} &= \bigl(1 + \gamma_m(\alpha_{\rm IR})\bigr)(m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s) \\ &\to 0\ ,\ SU(3)_L \times SU(3)_R \text{ limit} \end{aligned} \quad (2)$$

and hence a 0^{++} QCD dilaton[2] σ due to quark condensation.[3] The obvious candidate for this state is the $f_0(500)$, which arises from a pole on the second sheet at a complex mass with typical value [1]

$$m_{f_0} = 441 - i\,272 \text{ MeV} \quad (3)$$

and surprisingly small errors [19]. In all estimates of this type, the real part of m_{f_0} is less than m_K.

In Sec. II below, we recall problems with the phenomenology of χPT$_3$ caused by the f_0 pole in 0^{++} channels, and observe that they can be avoided by treating f_0 as a Nambu-Goldstone (NG) boson σ in the limit (2). The result is chiral-scale perturbation theory χPT$_\sigma$, where the NG sector $\{\pi, K, \eta, f_0/\sigma\}$ is clearly separated in scale from other hadrons.

Section III introduces the model-independent χPT$_\sigma$ Lagrangian for meson amplitudes expanded in α_s about $\alpha_{\rm IR}$ for $m_{u,d,s} \sim 0$. It summarizes soft π, K, η, σ meson theorems for three-flavor chiral and scale symmetry. For amplitudes where σ plays no role, the results agree with χPT$_3$. Results for soft σ amplitudes (Sec. IV) are similar to those found originally [5, 6] but include effects due to the gluonic term in (1). In Appendix A, Weinberg's

*Electronic address: rcrewthe@physics.adelaide.edu.au
†Electronic address: tunstall@itp.unibe.ch

[1] We have $[D_\mu\,,\,D_\nu] = igG^a_{\mu\nu}T^a$ where D_μ is the covariant derivative, $\{T^a\}$ generate the gauge group, $\alpha_s = g^2/4\pi$ is the strong coupling, and $\beta = \mu\partial\alpha_s/\partial\mu$ and $\gamma_m = \mu\partial \ln m_q/\partial\mu$ refer to a mass-independent renormalization scheme with scale μ.

[2] We reserve the term *dilaton* and notation σ for a NG boson due to scale invariance being preserved by the Hamiltonian but broken by the vacuum, in some limit. We are *not* talking about the σ-model, scalar gluonium [8], or walking gauge theories [9–12] where $\beta \approx 0$ near a scale-invariant vacuum [13–15] and proposals for "dilatons" [12, 16, 17] seem unlikely [18].

[3] In field and string theory, it is often stated that Green's functions are manifestly conformal invariant for $\beta = 0$. This assumes that, as in perturbative theories with $\beta = 0$, there are no scale condensates. If a scale condensate is present, conformal invariance becomes manifest only if *all* four-momenta are spacelike and large.

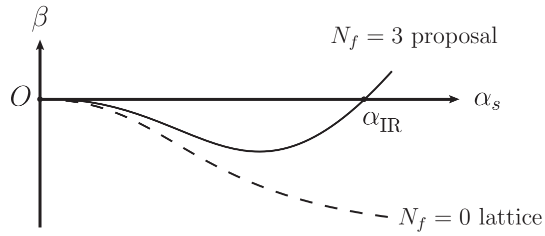

FIG. 1: The solid line shows a three-flavor β function (or better, a QCD version [20] of the Gell-Mann–Low Ψ function) with an infrared fixed point $\alpha_{\rm IR}$ at which α_s freezes [21–26] but the manifest scale invariance of [13–15] is *avoided*. The existence of $\alpha_{\rm IR}$ for small N_f values is not entirely settled. The dashed line shows the original lattice result [27] for $N_f = 0$ (no quarks) where β remains negative and becomes linear at large α_s.

analysis of the χPT$_2$ loop expansion [28] is extended to include χPT$_\sigma$.

Effective electromagnetic and weak operators are then added to simulate two-photon processes (Sec. VI) and nonleptonic K decays (Sec. VII). The main result is a simple explanation of the $\Delta I = 1/2$ rule for kaon decays: in the *lowest* order of χPT$_\sigma$, there is a dilaton pole diagram (Fig. 2) which produces most of the $\{\pi\pi\}_{I=0}$ amplitude

$$A_0 = A_{g_{8,27}\,\text{vertices}} + A_{\sigma\text{-pole}} \simeq A_{\sigma\text{-pole}} \tag{4}$$

and makes it large relative to the $I = 2$ amplitude A_2 [3]:

$$\big|A_0/A_2\big|_{\rm expt} \simeq 22\,. \tag{5}$$

We conclude that the ratio of the **8** and **27** contact couplings g_8 and g_{27} is of the order

$$1 \lesssim \big|g_8/g_{27}\big| \lesssim 5 \tag{6}$$

indicated by early calculations [29–32], and *not* the value 22 found by fitting lowest order χPT$_3$ to data.

In order to obtain a value for the $K_S\sigma$ coupling of Fig. 2, we compare the two-photon processes $\gamma\gamma \to \pi\pi$ (Fig. 3) and $K_S \to \gamma\gamma$. Well-known features of these amplitudes are the presence of ultraviolet finite $\pi^\pm, K^\pm$ loop diagrams coupled to the external photons [33], and the need for a rule [34–36]

$$A_\mu \sim \partial_\mu = O(p) \tag{7}$$

specifying the effect of a photon or weak boson field A_μ on the chiral order of terms in loop expansions. These features are important for our analysis, and in particular, for an investigation in Sec. VI of the relation between the $\sigma\gamma\gamma$ coupling (Fig. 3) and the electromagnetic trace anomaly [37, 38]

$$\begin{aligned} \widetilde{\theta}^\mu_\mu &= \theta^\mu_\mu + (R\alpha/6\pi)F_{\mu\nu}F^{\mu\nu}, \\ R &= \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\bigg|_{\text{energy}\to\infty} \end{aligned} \tag{8}$$

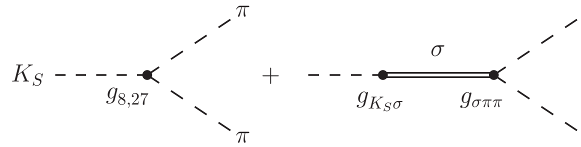

FIG. 2: Tree diagrams in the effective theory χPT$_\sigma$ for the decay $K_S \to \pi\pi$. The vertex amplitudes due to **8** and **27** contact couplings g_8 and g_{27} are dominated by the σ/f_0 pole amplitude. The magnitude of $g_{K_S\sigma}$ is found by applying χPT$_\sigma$ to $K_S \to \gamma\gamma$ and $\gamma\gamma \to \pi\pi$.

at the QCD infrared fixed point $\alpha_s = \alpha_{\rm IR}$. Here $F_{\mu\nu}$ and α are the electromagnetic field strength tensor and fine-structure constant, and $\widetilde{\theta}_{\mu\nu}$ is the energy-momentum tensor for QCD and QED combined.

To obtain an approximate result for the decay $\sigma \to \gamma\gamma$, the momentum q carried by θ^μ_μ has to be extrapolated from $q^2 = 0$ (given exactly by the electromagnetic trace anomaly) to $q^2 = m_\sigma^2$. In simple cases, and when photons are absent, this amounts to σ-pole dominance of θ^μ_μ, i.e. partial conservation of the dilatation current (PCDC) [39], which is the direct analogue of partial conservation of the axial-vector current (PCAC) for soft-pion amplitudes. However, we find that, unlike PCAC for $\pi^0 \to \gamma\gamma$, PCDC for $\sigma \to \gamma\gamma$ is modified by meson loop diagrams coupled to photons. In effect, these ultraviolet convergent diagrams produce an infrared singularity which is an inverse *power* of the light quark mass, arising in the same way as conventional Li-Pagels singularities [40, 41], but sufficiently singular to compete with the pole term.

In Appendix B, we show that, for a fixed number of external operators coupled purely to the NG sector, these inverse-power singularities do not upset the convergence of the chiral expansion: relative to the corresponding lowest order graph, be it tree or loop, each additional loop produces a factor $O(m_q)$ or $O(m_q \ln m_q)$. The analysis generalises the rule (7) for minimal gauge couplings [34, 35] and its extension to axial anomalies [36] to include (a) other nonminimal gauge couplings such as the electromagnetic trace anomaly (8), and (b) external Wilson operators of any kind. Appendix C is a brief note about Eq. (8) for QCD in the physical region $0 < \alpha_s < \alpha_{\rm IR}$.

Unlike other results in this article, our estimate

$$R_{\rm IR} \approx 5 \tag{9}$$

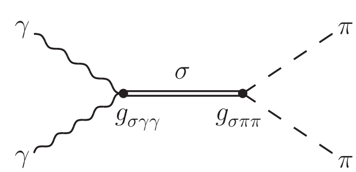

FIG. 3: Dilaton pole in $\gamma\gamma \to \pi\pi$. In this order of χPT$_\sigma$, diagrams with a $\pi^\pm$ or $K^\pm$ loop coupled to both photons must also be included.

for the renormalized value of R at the fixed point depends on the many-color limit $N_c \to \infty$. This involves the observation (Sec. II) that for N_c large, the dilaton σ/f_0 is a $q\bar{q}$ state, i.e. similar to π, K, η, but with planar-gluon corrections. Like other $q\bar{q}$ resonances, σ/f_0 has a *narrow* width in that limit (Sec. IV).

II. MOTIVATION

It may seem odd that new conclusions about QCD can be drawn simply from approximate chiral symmetry and 0^{++} pole diagrams. Scalar pole dominance for reactions like $K_S \to \pi\pi$ was considered long ago [42–45], it can be easily incorporated in a chiral invariant way, and if difficulties with hyperon decays[4] are overlooked, theory and experiment for soft π, K, η amplitudes are in excellent agreement, with dispersive corrections included where necessary.

The flaw in this picture is contained in another old observation — lowest order χPT_3, if not corrected, typically fails for amplitudes which involve both a 0^{++} channel and $O(m_K)$ extrapolations in momenta:

1. Final-state $\pi\pi$ interactions [46] in $K_{\ell 4}$ decays [47] and nonleptonic K [48, 49] and η [50, 51] decays compete with and often dominate purely chiral contributions [46–52].

2. The chiral one-loop prediction for the $K_L \to \pi^0\gamma\gamma$ rate [53] is only 1/3 of the measured value [54].

3. The lowest order prediction [55, 56] of a linear rise in the $\gamma\gamma \to \pi^0\pi^0$ cross section disagrees [57] with the Crystal Ball data [58].

These facts became evident at a time when it was thought that 0^{++} resonances below ≈ 1 GeV did not exist,[5] but it was already clear that agreement with data required the inclusion of large dispersive effects which had to be somehow "married" to chiral predictions [61]. The same can be said now, except that the $f_0(500)$ pole of Eq. (3) can be identified as the source of these effects. Consequently dispersion theory for these processes, with the possible exception of $\eta \to 3\pi$ decay [62], is far better understood [45, 63–66].

But that does nothing to alter the fact that the lowest order of standard chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT_3 fits these data so poorly. The lowest order amplitude $\mathcal{A}_{\text{LO}}$ is the first term of an asymptotic series

$$\mathcal{A} = \big\{\mathcal{A}_{\text{LO}} + \mathcal{A}_{\text{NLO}} + \mathcal{A}_{\text{NNLO}} + \ldots\big\}_{\chi\text{PT}_3} \tag{10}$$

in powers of $O(m_K)$ momenta and quark masses $m_{u,d,s} = O(m_K^2)$ (with $m_{u,d}/m_s$ held fixed). If the first term is a poor fit, *any* truncation of the series to make it agree with a dispersive fit to data is unsatisfactory *because the series is diverging.*

For example, consider the amplitude for $K_L \to \pi^0\gamma\gamma$ (item 2 above). Let the series (10) be matched to data by including dispersive NLO corrections (next to lowest order) and then truncating:

$$\mathcal{A}_{K_L\to\pi^0\gamma\gamma} \simeq \big\{\mathcal{A}_{\text{LO}} + \mathcal{A}_{\text{NLO}}\big\}_{\chi\text{PT}_3}\,. \tag{11}$$

The LO prediction for the rate is 1/3 too small, so, depending on the relative phase of the LO and NLO terms, a fit can be achieved only for

$$\big|\mathcal{A}_{\text{NLO}}\big|_{\chi\text{PT}_3} \gtrsim \sqrt{2}\big|\mathcal{A}_{\text{LO}}\big|_{\chi\text{PT}_3}\,. \tag{12}$$

How can this be reconciled with the success [35] of χPT_3 elsewhere? Corrections to lowest order χPT_3 should be $\sim 30\%$ at most:

$$\big|\mathcal{A}_{\text{NLO}}/\mathcal{A}_{\text{LO}}\big|_{\chi\text{PT}_3} \lesssim 0.3\ ,\ \text{acceptable fit.} \tag{13}$$

A standard response[6] is that there are limits to the applicability of an expansion like χPT_3, so failures in a few cases are to be expected.

In our view, there is a consistent trend of failure in 0^{++} channels which can and should be corrected by modifying the *lowest order* of the three-flavor theory. This must be achieved without changing χPT_2, where amplitudes are expanded about the chiral $SU(2)_L \times SU(2)_R$ limit with $O(m_\pi)$ extrapolations[7] in momenta; χPT_2 is wholly successful, producing convergent results with small corrections, typically 5% or at most 10%:

$$\big|\mathcal{A}_{\text{NLO}}/\mathcal{A}_{\text{LO}}\big|_{\chi\text{PT}_2} < 0.1\ ,\ \text{observed fits.} \tag{14}$$

Our solution is to replace χPT_3 by chiral-scale perturbation theory χPT_σ, whose NG sector $\{\pi, K, \eta, \sigma/f_0\}$ includes $f_0(500)$ as a dilaton σ associated with the scale-invariant limit (2). In χPT_σ, the strange quark mass m_s sets the scale of $m_{f_0}^2$ as well as m_K^2 and m_η^2 (Fig. 4, bottom diagram). As a result, the rules for counting powers of m_K are changed: f_0 pole amplitudes (NLO in χPT_3) are promoted to LO. That fixes the LO problem for amplitudes involving 0^{++} channels and $O(m_K)$ extrapolations in momenta. At the same time, χPT_σ *preserves* the LO successes of χPT_3 elsewhere: for reactions which do not involve σ/f_0, the predictions of χPT_3 and χPT_σ are identical.

The analysis relies on a clear distinction being drawn between χPT_2, χPT_3, and χPT_σ. For each amplitude

---

[4] Accounting for nonleptonic hyperon decays will require either χPT for baryons or the weak sector of the Standard Model to be modified.

[5] The $\epsilon(700)$ resonance considered in [5, 6, 37, 38] was last listed in 1974 [59]. Replacing it by $f_0(500)$ was proposed in 1996 [60].

[6] LCT thanks Professor H. Leutwyler for a discussion of this point.

[7] For some authors, "two-flavor theory" refers to pionic processes *without* the restriction $O(m_\pi)$ on pion momenta. Then the relevant theory is χPT_3 or χPT_σ, not χPT_2. See Fig. 4.

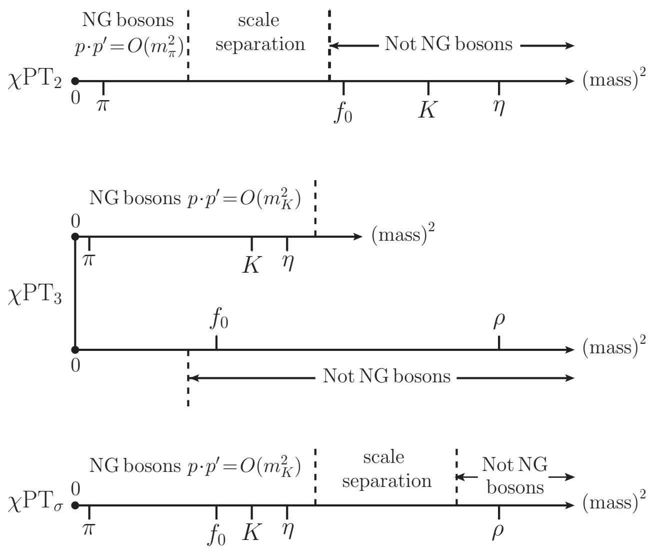

FIG. 4: Scale separations between Nambu-Goldstone (NG) sectors and other hadrons for each type of chiral perturbation theory χPT discussed in this paper. Note that scale separation in χPT_2 (chiral $SU(2) \times SU(2)$, top diagram) is ensured by limiting extrapolations in momenta p, p' to $O(m_\pi)$ (not $O(m_K)$). In conventional three-flavor theory χPT_3 (middle diagram), there is *no scale separation*: the non-NG boson $f_0(500)$ sits in the middle of the NG sector $\{\pi, K, \eta\}$. Our three-flavor proposal χPT_σ (bottom diagram) for $O(m_K)$ extrapolations in momenta implies a clear scale separation between the NG sector $\{\pi, K, \eta, \sigma = f_0\}$ and the non-NG sector $\{\rho, \omega, K^*, N, \eta', \ldots\}$.

$\mathcal{A}$, these three versions of χPT produce three inequivalent asymptotic expansions of the form (10). The corresponding scale separations between NG sectors and other particles are shown in Fig. 4.

We use χPT_2 in the strict sense originally intended [34, 41, 67–69]: an asymptotic expansion for the limit $m_{u,d} \to 0$ with $m_s \neq 0$ and (crucially) momentum extrapolations limited to $O(m_\pi)$. There are only three NG bosons $\{\pi^+, \pi^0, \pi^-\}$, with *no dilaton*: χPT_2 is not sensitive to the behavior of β because of the relatively large term $m_s \bar{s}s$ in Eq. (1) for θ^μ_μ. Since s is not treated as a light quark, the K and η mesons as well as $f_0, \rho, \omega, N, \eta' \ldots$ are excluded from the χPT_2 NG sector.

If there is an $O(m_K)$ extrapolation in momentum, χPT_2 is *not* sufficient. Three-flavor contributions must be included, either as large dispersive extrapolations, or with χPT_2 replaced by a three-flavor chiral expansion: χPT_3 [35, 41, 70–72] or χPT_σ.

An $O(m_K)$ extrapolation may arise because K or η is soft, or because the pion momenta in (say) $\pi\pi \to \pi\pi$ or $\gamma\gamma \to \pi\pi$ are chosen to be $O(m_K)$, or because of a kinematic constraint. A well known example is the fact that χPT_2 says almost nothing about $K_S \to \pi\pi$: if one pion becomes soft, the momentum difference between on-shell states $|K\rangle$ and $|\pi\rangle$ is necessarily $O(m_K)$. An example of interest in Sec. VII is the pion-loop result [33] for $K_S \to \gamma\gamma$, which is not implied by χPT_2: a three-flavor expansion is necessary.

Both χPT_3 and χPT_σ involve the limit[8]

$$m_i \sim 0\ ,\ m_i/m_j \text{ fixed}, i, j = u, d, s. \tag{15}$$

In each case, amplitudes are expanded in powers and logarithms of

$$\{\text{momenta}\}/\chi_{\text{ch}} \ll 1 \tag{16}$$

where the infrared mass scale $\chi_{\text{ch}} \approx 1$ GeV is set by the chiral condensate $\langle \bar{q}q \rangle_{\text{vac}}$. In χPT_3, χ_{ch} is $4\pi F_\pi$ [75], where $F_\pi = 93$ MeV is the pion decay constant; a similar result will be found for χPT_σ in Sec. IV. The chiral scale χ_{ch} also sets the mass scale of particles outside the corresponding NG sectors.[9] For nucleons with mass M_N, this is evident from the Goldberger-Treiman relation

$$F_\pi g_{\pi NN} \simeq g_A M_N\ . \tag{17}$$

It is essential [75] to make a clear distinction between the low-energy scale χ_{ch} and the ultraviolet QCD scale $\Lambda_{\text{QCD}} \approx 200$ MeV associated with expansions in the asymptotically free domain

$$\{\text{momenta}\}/\Lambda_{\text{QCD}} \gg 1. \tag{18}$$

Strong gluonic fields are presumably responsible for both scales, but that does not mean that the dimensionless ratio

$$\chi_{\text{ch}}/\Lambda_{\text{QCD}} \approx 5 \tag{19}$$

has to be 1.

The difference between χPT_3 and χPT_σ can be seen in the relation between hadronic masses and terms in Eq. (1) for θ^μ_μ.

In χPT_3, there is no sense in which the gluonic trace anomaly is small. For example, the gluonic anomaly is taken to be responsible for most of the nucleon's mass:

$$M_N = \langle N|\theta^\mu_\mu|N\rangle \underset{\chi\text{PT}_3}{=} \frac{\beta(\alpha_s)}{4\alpha_s}\langle N|G^2|N\rangle + O\big(m_K^2\big)\ . \tag{20}$$

This assumes that $f_0(500)$ pole terms can be neglected, or equivalently, given that f_0 is so light on the mass scale for non-NG particles set by χ_{ch}, that f_0 couples weakly to G^2 and $\bar{q}q$. As noted in Fig. 4, the small f_0 mass implies that χPT_3 has no scale separation, which (as we have seen) is a problem because f_0 couples so strongly to other particles.

[8] We require $m_s > m_{u,d}$ throughout. Double asymptotic series can be considered for either χPT_2 and χPT_3 [35, 73] or χPT_2 and χPT_σ. The unusual limit $m_s \to 0$ for fixed $m_{u,d} \neq 0$ considered in Sec. 4 of [74] does not produce any NG bosons.

[9] Except for glueballs, if they exist. In χPT_σ, they may have large masses due to gluonic scale condensates such as $\langle G^2 \rangle_{\text{vac}}$.

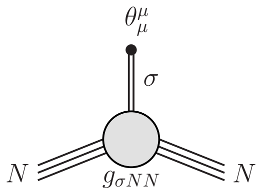

FIG. 5: Dominant σ-pole diagram in $\chi\mathrm{PT}_\sigma$ for $\langle N|\theta^\mu_\mu|N\rangle$.

Contrast this with $\chi\mathrm{PT}_\sigma$, where the infrared regime

$$O(m_K)\ \text{momenta}\ \ll \chi_{\mathrm{ch}} \tag{21}$$

emphasizes values of α_s close to α_{IR}, so a combined limit

$$m_{u,d,s} \sim 0 \quad \text{and} \quad \alpha_s \lesssim \alpha_{\mathrm{IR}} \tag{22}$$

must be considered. Since $\beta(\alpha_s)$ is small, the gluonic trace anomaly is small *as an operator*, but it can produce large amplitudes when coupled to dilatons.

Consider how M_N arises in $\chi\mathrm{PT}_\sigma$ (Fig. 5). Like other pseudo-NG bosons, σ couples to the vacuum via the divergence of its symmetry current,

$$\langle\sigma|\theta^\mu_\mu|\mathrm{vac}\rangle = -m_\sigma^2 F_\sigma = O(m_\sigma^2)\,,\ m_\sigma \to 0 \tag{23}$$

where F_σ is the dilaton decay constant. The nucleon remains massive in the scaling limit because of its coupling $-g_{\sigma NN}\sigma\bar{N}N$ to σ and the factor $-i/m_\sigma^2$ produced by the σ pole at zero momentum transfer. This gives rise to the well known analogue [39]

$$F_\sigma g_{\sigma NN} \simeq M_N \tag{24}$$

of the Goldberger-Trieman relation (17).

In our scheme, both the gluonic anomaly and the quark mass term in Eq. (1) for θ^μ_μ can contribute to M_N in the chiral-scale limit (2). That is because we require[10]

$$m_\sigma^2 = O(m_K^2) = O(m_{u,d,s})\,, \tag{25}$$

which allows the constants F_{G^2} and $F_{\bar{q}q}$ given by

$$\begin{aligned} \beta(\alpha_s)/(4\alpha_s)\langle\sigma|G^2|\mathrm{vac}\rangle &= -m_\sigma^2 F_{G^2}\,, \\ \{1+\gamma_m(\alpha_s)\}\sum_{q=u,d,s} m_q\langle\sigma|\bar{q}q|\mathrm{vac}\rangle &= -m_\sigma^2 F_{\bar{q}q} \end{aligned} \tag{26}$$

to remain finite in that limit:

$$M_N \simeq F_{G^2} g_{\sigma NN} + F_{\bar{q}q} g_{\sigma NN}\,. \tag{27}$$

[10] In principle, we could have constructed a chiral-scale perturbation theory with m_σ and m_K as independent expansion parameters, but that would make sense only if there were a fourth light quark or different low-energy scales for chiral and scale expansions. Fig. 4 provides clear confirmation that the choice $m_\sigma = O(m_K)$ is sensible.

Suggestions that a resonance like $f_0(500)$ cannot be a pseudo-NG boson have no foundation. There can be no theorem to that effect because counterexamples such as our effective chiral-scale Lagrangian in Sec. III are so easily constructed. It is true that in the symmetry limit where a NG boson becomes exactly massless, it has zero width, but that is because there is no phase space for it to decay into other massless particles. If phase space for strong decay is made available by explicit symmetry breaking and quantum number conservation allows it, a pseudo-NG boson will decay:

$$m_\sigma > 2m_\pi \ \Rightarrow\ \text{width } \Gamma_{\sigma\to\pi\pi} \neq 0\,. \tag{28}$$

Note that:

- Non-NG bosons need not be resonances; for example, $\eta'(960)$ is stable against strong decay.
- The resonance f_0/σ becomes a massless NG boson *only* if all three quarks u, d, s become massless as α_s tends to α_{IR}. In that combined limit, all particles except π, K, η and σ remain massive. Strong gluon fields set the scale of the condensate $\langle\bar{q}q\rangle_{\mathrm{vac}}$, which then sets the scale for massive particles and resonances except (possibly) glueballs.
- QCD at $\alpha_s = \alpha_{\mathrm{IR}}$ resembles the physical theory (i.e. QCD for $0 < \alpha_s < \alpha_{\mathrm{IR}}$) in the resonance region, but differs completely at high energies because it lacks asymptotic freedom. Instead, Green's functions scale asymptotically with nonperturbative anomalous dimensions in the ultraviolet limit.

Another key difference between $\chi\mathrm{PT}_3$ and $\chi\mathrm{PT}_\sigma$ becomes evident in the many-color limit $N_c \to \infty$ [76–78]. At issue is the quark content of the $f_0(500)$ resonance: is it a standard $q\bar{q}$ meson, or an exotic tetraquark state $q\bar{q}q\bar{q}$? In general, this is a model-dependent question; indeed the tetraquark idea was first proposed for the 0^+ nonet in the context of the quark-bag model [79]. However the large-N_c limit permits conclusions which are far less model-dependent.

In modern analyses of $\chi\mathrm{PT}_3$, $f_0(500)$ is often considered to be a multi-particle state and so is not represented by a field in an effective Lagrangian. Instead, the $\chi\mathrm{PT}_3$ expansion is unitarized, with f_0 identified as a resonating two-meson state produced by the unitarized structure. From that, the large-N_c conclusion [80]

$$f_0 \sim \pi\pi \sim (q\bar{q})^2\,,\ \ \text{unitarized } \chi\mathrm{PT}_3 \tag{29}$$

is drawn. This assumes from the outset that f_0 is *not* a dilaton. The problem, already discussed at the beginning of this Section, is that the $\chi\mathrm{PT}_3$ expansion diverges because it is dominated by these unitary "corrections".

In $\chi\mathrm{PT}_\sigma$, the large-N_c properties of f_0/σ are similar to those of pions, and are found by considering the two-point function of $\theta_{\mu\nu}$ instead of chiral currents. At large-N_c, the spin-2 part is dominated by pure-glue states:

$$T\big\langle\mathrm{vac}\big|\theta_{\alpha\beta}\theta_{\mu\nu}\big|\mathrm{vac}\big\rangle_{\mathrm{spin-2}} = O(N_c^2)\,. \tag{30}$$

However, when the spin-0 part is projected out by taking the trace θ^α_α, the quark term dominates the gluonic anomaly of Eq. (1) at large N_c because of the factor $\alpha_s \sim 1/N_c$ multiplying G^2. Thus we find

$$T\big\langle \mathrm{vac}\big|\theta^\alpha_\alpha \theta^\mu_\mu\big|\mathrm{vac}\big\rangle = O(N_c) \tag{31}$$

due to the quark term compared with $O(1)$ from the gluonic anomaly. Clearly, a σ pole can be present only if f_0/σ is a $q\bar{q}$ state. At zero momentum transfer, this pole contributes $m_\sigma^2 F_\sigma^2$ to the amplitude (31), from which we conclude

$$F_\sigma = O\big(\sqrt{N_c}\big)\,, \tag{32}$$

as for the pion decay constant F_π. We will see in Sec. IV that the dilaton, like other $q\bar{q}$ states, obeys the narrow width rule at large N_c.

Sometimes pure-glue corrections in f_0/σ are dominant. The most obvious example is the nucleon mass M_N, where the leading $O(N_c)$ contribution due to $q\bar{q}$ states is the numerically small two-flavor sigma term

$$\langle N|m_u\bar{u}u + m_d\bar{d}d|N\rangle \ll M_N\,. \tag{33}$$

Therefore (as is generally agreed) most of M_N comes from the $m_{u,d}$-independent term due to pure-glue exchange. In particular, the terms $\sim G^2$ and $m_s\bar{s}s$ in Eq. (1) for θ^μ_μ couple to a nucleon only via pure-glue states.

III. CHIRAL-SCALE LAGRANGIAN

Consider strong interactions at low energies $\alpha_s \lesssim \alpha_{\mathrm{IR}}$ within the physical region

$$0 < \alpha_s < \alpha_{\mathrm{IR}}\,. \tag{34}$$

Let d denote the scaling dimension of operators used to construct an effective chiral-scale Lagrangian. In general, there must be a scale-invariant term $\mathcal{L}_{\mathrm{inv}}$ with scaling dimension $d = 4$, a term $\mathcal{L}_{\mathrm{mass}}$ with dimension [81]

$$d_{\mathrm{mass}} = 3 - \gamma_m\big(\alpha_{\mathrm{IR}}\big)\;, \quad 1 \leqslant d_{\mathrm{mass}} < 4 \tag{35}$$

to simulate explicit breaking of chiral symmetry by the quark mass term, and a term $\mathcal{L}_{\mathrm{anom}}$ with dimension $d > 4$ to account for gluonic interactions responsible for the strong trace anomaly in Eq. (1):

$$\mathcal{L}_{\chi\mathrm{PT}_\sigma} = :\mathcal{L}^{d=4}_{\mathrm{inv}} + \mathcal{L}^{d>4}_{\mathrm{anom}} + \mathcal{L}^{d<4}_{\mathrm{mass}} :\;. \tag{36}$$

The anomalous part of d_{mass} is evaluated at α_{IR} because we expand in α_s about α_{IR}. A proof that $\mathcal{L}_{\mathrm{anom}}$ has dimension $d > 4$ appears later in this Section.

We restrict our analysis to the NG sector of $\chi\mathrm{PT}_\sigma$ (Fig. 4). Then operators in

$$\mathcal{L}_{\chi\mathrm{PT}_\sigma} = \mathcal{L}\big[\sigma, U, U^\dagger\big] \tag{37}$$

are constructed from a QCD dilaton field σ and the usual chiral $SU(3)$ field

$$U = U(\pi, K, \eta)\;,\; UU^\dagger = I\,. \tag{38}$$

Scale and chiral transformations commute, so σ is chiral invariant. The scale dimensions of π, K, η and hence U must be zero in order to preserve the range of field values on the coset space $SU(3)_L \times SU(3)_R/SU(3)_V$ [82].

In Eq. (36), both $\mathcal{L}_{\mathrm{inv}}$ and $\mathcal{L}_{\mathrm{anom}}$ are $SU(3)_L \times SU(3)_R$ invariant, while $\mathcal{L}_{\mathrm{mass}}$ belongs to the representation $(\mathbf{3}, \bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}}, \mathbf{3})$ associated with the π, K, η (mass)^2 matrix M. In lowest order, with M diagonalized,

$$M = \frac{F_\pi^2}{4}\begin{pmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2m_K^2 - m_\pi^2 \end{pmatrix} \tag{39}$$

the vacuum condition for U is

$$U \to I \quad \text{for} \quad \pi, K, \eta \to 0\,. \tag{40}$$

The dimension of $\mathcal{L}_{\mathrm{anom}}$ can be found from the scaling Ward identities (Callan-Symanzik equations)

$$\bigg\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \gamma_m(\alpha_s)\sum_q m_q\frac{\partial}{\partial m_q}\bigg\}\mathcal{A} = 0 \tag{41}$$

for renormalization-group invariant QCD amplitudes $\mathcal{A}$. The term $\beta\partial/\partial\alpha_s$ corresponds to the gluonic anomaly in Eq. (1), so the effect of $\alpha_s\partial/\partial\alpha_s$ on $\mathcal{A}$ is to insert the QCD operator $G^2 = G^a_{\mu\nu}G^{a\mu\nu}$ at zero momentum transfer. Applying $\alpha_s\partial/\partial\alpha_s$ to Eq. (41),

$$\begin{aligned}\bigg\{\mu\frac{\partial}{\partial\mu}+\beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \beta'(\alpha_s) - \beta(\alpha_s)/\alpha_s\bigg\}\alpha_s\frac{\partial\mathcal{A}}{\partial\alpha_s} \\ = -\alpha_s\frac{\partial}{\partial\alpha_s}\sum_q \gamma_m(\alpha_s)m_q\frac{\partial\mathcal{A}}{\partial m_q}\end{aligned} \tag{42}$$

we see that the anomalous dimension function for G^2 is

$$\gamma_{G^2}(\alpha_s) = \beta'(\alpha_s) - \beta(\alpha_s)/\alpha_s\,. \tag{43}$$

Hence, to lowest order in the expansion $\alpha_s \lesssim \alpha_{\mathrm{IR}}$, $\mathcal{L}_{\mathrm{anom}}$ has a positive anomalous dimension equal to the slope of β at the fixed point (Fig. 1):

$$d_{\mathrm{anom}} = 4 + \beta'\big(\alpha_{\mathrm{IR}}\big) > 4\,. \tag{44}$$

As $\alpha_s \to \alpha_{\mathrm{IR}}$, the gluonic anomaly vanishes, so for consistency,[10] we must require terms in $\mathcal{L}_{\mathrm{anom}}$ to involve derivatives $\partial\partial = O(M)$ or have $O(M)$ coefficients:

$$\mathcal{L}_{\mathrm{anom}} = O(\partial^2, M)\,. \tag{45}$$

The result is a chiral-scale perturbation expansion $\chi\mathrm{PT}_\sigma$ about α_{IR} with QCD dilaton mass $m_\sigma = O(m_K)$.

An explicit formula for the $\chi\mathrm{PT}_\sigma$ Lagrangian (36) can be readily found by following the approach of Ellis [5, 83]. Let F_σ be the coupling of σ to the vacuum via the energy momentum tensor $\theta_{\mu\nu}$, improved [84] when spin-0 fields are present:

$$\langle\sigma(q)|\theta_{\mu\nu}|\text{vac}\rangle = (F_\sigma/3)\big(q_\mu q_\nu - g_{\mu\nu}q^2\big)\,. \qquad (46)$$

When conformal symmetry is realized nonlinearly [85], a dilaton field σ is needed to create connection terms $\sim \partial\sigma$ in covariant derivatives. It transforms as

$$\sigma \to \sigma - \tfrac{1}{4}F_\sigma \log\big|\det(\partial x'/\partial x)\big| \qquad (47)$$

under conformal transformations $x \to x'$, which corresponds to scale dimension 1 for the covariant field e^{σ/F_σ}. The dimensions of χPT_3 Lagrangian operators such as

$$\mathcal{K}\big[U,U^\dagger\big] = \tfrac{1}{4}F_\pi^2\text{Tr}(\partial_\mu U\partial^\mu U^\dagger) \qquad (48)$$

and the dilaton operator $\mathcal{K}_\sigma = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma$ can then be adjusted by powers of e^{σ/F_σ} to form terms in $\mathcal{L}$. In lowest order,

$$\begin{aligned}
&\mathcal{L}^{d=4}_{\text{inv, LO}} = \big\{c_1\mathcal{K} + c_2\mathcal{K}_\sigma + c_3 e^{2\sigma/F_\sigma}\big\}e^{2\sigma/F_\sigma}\,,\\
&\mathcal{L}^{d>4}_{\text{anom, LO}}\\
&= \big\{(1-c_1)\mathcal{K} + (1-c_2)\mathcal{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\big\}e^{(2+\beta')\sigma/F_\sigma}\,,\\
&\mathcal{L}^{d<4}_{\text{mass, LO}} = \text{Tr}(MU^\dagger + UM^\dagger)e^{(3-\gamma_m)\sigma/F_\sigma}\,, \qquad (49)
\end{aligned}$$

where β' and γ_m are the anomalous dimensions $\beta'(\alpha_{\text{IR}})$ and $\gamma_m(\alpha_{\text{IR}})$ of Eqs. (44) and (35).

The constants c_1 and c_2 are not fixed by general arguments, while c_3 and c_4 depend on the vacuum condition chosen for the field σ. The role of c_3 and c_4 is to fix the scale of e^{σ/F_σ}, just as the (mass)^2 matrix fixes the chiral $SU(3)$ direction of U (Eqs. (39) and (40)). The simplest choice of field variables[11] is to have all NG fields σ, π, K, η fluctuate about zero.

For the vacuum to be stable in the σ direction at $\sigma = 0$, Lagrangian terms linear in σ must cancel:

$$\begin{aligned}
4c_3 + (4+\beta')c_4 &= -(3-\gamma_m)\big\langle\text{Tr}(MU^\dagger + UM^\dagger)\big\rangle_{\text{vac}}\\
&= -(3-\gamma_m)F_\pi^2\big(m_K^2 + \tfrac{1}{2}m_\pi^2\big)\,. \qquad (50)
\end{aligned}$$

Eqs. (45) and (50) imply that both c_3 and c_4 are $O(M)$.

Evidently χPT_σ is a simple extension of the conventional three-flavor theory χPT_3. The χPT_σ Lagrangian defined by Eqs. (36) and (49) satisfies the condition

$$\mathcal{L}_{\chi\text{PT}_\sigma} \to \mathcal{L}_{\chi\text{PT}_3}\,,\ \sigma \to 0 \qquad (51)$$

and hence preserves the phenomenological success of *lowest order* χPT_3 for amplitudes which do not involve the 0^{++} channel (Sec. II). In next to lowest order, new chiral-scale loop diagrams involving σ need to be checked.

[11] On-shell amplitudes do not depend on how the field variables are chosen [86, 87].

The χPT_σ Lagrangian obeys the standard rule that each term $\mathcal{L}_d$ of dimension d contributes $(d-4)\mathcal{L}_d$ to the trace of the effective energy-momentum tensor:

$$\theta^\mu_\mu\big|_{\text{eff}} = :\beta'\mathcal{L}^{d>4}_{\text{anom}} - (1+\gamma_m)\mathcal{L}^{d<4}_{\text{mass}}:\,. \qquad (52)$$

Note that the critical exponent β' normalizes the gluonic term in θ^μ_μ.

IV. STRONG INTERACTIONS

In lowest order, $\mathcal{L}$ gives formulas for the $\sigma\pi\pi$ coupling

$$\mathcal{L}_{\sigma\pi\pi} = \big\{\big(2+(1-c_1)\beta'\big)|\partial\boldsymbol{\pi}|^2 - (3-\gamma_m)m_\pi^2|\boldsymbol{\pi}|^2\big\}\sigma/(2F_\sigma) \qquad (53)$$

and dilaton mass m_σ

$$m_\sigma^2F_\sigma^2 = F_\pi^2\big(m_K^2 + \tfrac{1}{2}m_\pi^2\big)(3-\gamma_m)(1+\gamma_m) - \beta'(4+\beta')c_4 \qquad (54)$$

which resemble pre-QCD results [5, 6, 83, 88] but have extra gluonic terms proportional to β'. For consistency with data, we must assume that the unknown coefficient $2 + (1-c_1)\beta'$ in Eq. (53) does not vanish accidentally. That preserves the key feature of the original work, that $\mathcal{L}_{\sigma\pi\pi}$ is mostly *derivative*: for soft $\pi\pi$ scattering (energies $\sim m_\pi$), the dilaton pole amplitude is negligible because the $\sigma\pi\pi$ vertex is $O(m_\pi^2)$, while the $\sigma\pi\pi$ vertex for an on-shell dilaton

$$g_{\sigma\pi\pi} = -\big(2 + (1-c_1)\beta'\big)m_\sigma^2/(2F_\sigma) + O(m_\pi^2) \qquad (55)$$

is $O(m_\sigma^2)$, consistent with σ being the broad resonance $f_0(500)$.

Comparisons with data require an estimate of F_σ, most simply from NN scattering and the dilaton relation (24). The data imply [89] a mean value $g_{\sigma NN} \sim 9$ and hence $F_\sigma \sim 100\,\text{MeV}$ but with an uncertainty which is either model-dependent or very large ($\approx 70\%$). That accounts for the large uncertainty in

$$1\tfrac{1}{2} \lessapprox |2 + (1-c_1)\beta'| \lessapprox 6 \qquad (56)$$

when we compare Eq. (55) with data [1]:

$$|g_{\sigma\pi\pi}| = 3.31^{+0.35}_{-0.15}\ \text{GeV, and } m_\sigma \approx 441\,\text{MeV}. \qquad (57)$$

The convergence of a chiral-scale expansion can be tested by adding σ-loop diagrams to the standard χPT_3 analysis [35]. These involve the (as yet) undetermined constants $\beta', \gamma_m, c_{1\ldots4}$: for example, corrections to $g_{\sigma\pi\pi}$ involve the $\sigma\sigma\sigma$ and $\sigma\sigma\pi\pi$ vertices derived from Eq. (49).

However a numerical estimate of scales associated with the expansion can be obtained using the dimensional arguments of Manohar and Georgi [75]. The idea is to count powers of dimensionful quantities F_π and (for χPT_σ) F_σ associated with the quark condensate $\langle\bar{q}q\rangle_{\text{vac}}$, and keep track of powers of 4π arising from loop integrals. To illustrate their point, Manohar and Georgi considered loop

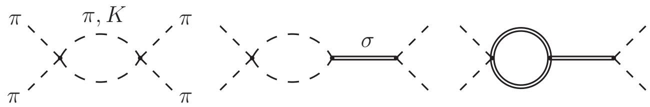

FIG. 6: Examples of NLO $\chi\mathrm{PT}_\sigma$ graphs in the chiral-scale expansion of $\pi\pi$ scattering for $O(m_K)$ momenta. Each vertex is generated by the lowest order terms (49) in $\mathcal{L}$. Not shown are additional diagrams involving the self-energy of the σ propagator, and internal σ lines which connect one external π leg to another. Similar diagrams are found for the t and u channels.

corrections to $\pi\pi$ scattering, such as the first diagram in Fig. 6, for which they obtained the estimate

$$\mathcal{A}_{\mathrm{loop}}/\mathcal{A}_{\mathrm{tree}} \sim \frac{1}{16\pi^2 F_\pi^2} \times \text{logarithms}. \tag{58}$$

In our scheme, we must add contributions

$$\sim \left\{ \frac{1}{16\pi^2 F_\sigma^2} \text{ and } \frac{F_\pi^2}{16\pi^2 F_\sigma^4} \right\} \times \text{logarithms} \tag{59}$$

from e.g. the second and third graphs of Fig. 6. As a result, we find that there are in principle *two* $\chi\mathrm{PT}_\sigma$ scales

$$\chi_\pi = 4\pi F_\pi \quad \text{and} \quad \chi_\sigma = 4\pi F_\sigma\,. \tag{60}$$

The rough estimate of 100 MeV for F_σ (close to $F_\pi \simeq 93$ MeV) indicates that in effect, there is a single infrared mass scale

$$\chi_\pi \approx \chi_\sigma \approx 1 \text{ GeV} \tag{61}$$

as foreshadowed in Eq. (16).

Numerology which ignores factors of 4π can be as misleading in $\chi\mathrm{PT}_\sigma$ as in $\chi\mathrm{PT}_3$. The most important example of this arises from the observation that $f_0(500)$ is almost as broad as it is heavy. Does this mean that the width of $f_0(500)$ is a *lowest* order effect, i.e. of the same order in m_σ as the real part of the mass? If so, would not that invalidate PCDC (partial conservation of the dilatation current), where dominance by a *real* pole is assumed for the lowest order?

To see that the answer is "no", let us estimate the σ width $\Gamma_{\sigma\pi\pi}$ in the spirit of Manohar and Georgi. We find

$$\Gamma_{\sigma\pi\pi} \approx \frac{|g_{\sigma\pi\pi}|^2}{16\pi m_\sigma} \sim \frac{m_\sigma^3}{16\pi F_\sigma^2} \sim 250 \text{ MeV} \tag{62}$$

so $\Gamma_{\sigma\pi\pi}$ is $O(m_\sigma^3)$ and hence *nonleading* relative to the mass m_σ. We are therefore justified in using just tree diagrams to generate the lowest order[12] of $\chi\mathrm{PT}_\sigma$, as in $\chi\mathrm{PT}_2$ and $\chi\mathrm{PT}_3$. (The main exception to this rule, for two-photon channels, is discussed in Sec. VI and Appendix B.) Pure numerology fails because F_σ in the denominator of (62) is an order of magnitude smaller than $\chi_{\pi,\sigma}$.

In the large-N_c limit, as shown in Sec. II, the dilaton behaves as a $q\bar{q}$ state. It follows that the gluonic corrections $\sim (1-c_1)\beta'$ in Eq. (55) for the $\sigma\pi\pi$ coupling correspond to disconnected quark diagrams, so they are nonleading

$$(1-c_1)\beta' = O\big(1/N_c\big) \tag{63}$$

and the pre-QCD result [5, 6]

$$F_\sigma g_{\sigma\pi\pi} \approx -m_\sigma^2 \tag{64}$$

is recovered for N_c large. It follows from Eq. (32) that σ decouples from $\pi\pi$ at large N_c:

$$g_{\sigma\pi\pi} = O\big(1/\sqrt{N_c}\big)\,. \tag{65}$$

Hence, like other $q\bar{q}$ states, the dilaton σ obeys the narrow width rule

$$\Gamma_{\sigma\pi\pi} = O(1/N_c)\,. \tag{66}$$

The technique used to obtain Eq. (49) from $\chi\mathrm{PT}_3$ works equally well for higher order terms in strong interactions, and also for external operators induced by electromagnetic or weak interactions (Sects. VI and VII).

In general, NLO terms in the strong interaction Lagrangian $\mathcal{L}$ are $O(\partial^4, M\partial^2, M^2)$. For example, let us construct $O(\partial^4)$ terms from the $\chi\mathrm{PT}_3$ operator $(\mathrm{Tr}\partial U \partial U^\dagger)^2$. It has dimension 4 already, so it appears unchanged in the scale-symmetric term

$$\mathcal{L}^{d=4}_{\mathrm{inv,\,NLO}} = \{\text{coefficient}\}(\mathrm{Tr}\partial U \partial U^\dagger)^2 + \ldots \tag{67}$$

i.e. without σ field dependence. The anomalous term has dimension greater than 4, so it depends on σ:

$$\mathcal{L}^{d>4}_{\mathrm{anom,\,NLO}} = \{\text{coefficient}\}\big(\mathrm{Tr}\partial U \partial U^\dagger\big)^2 e^{\beta'\sigma/F_\sigma}. \tag{68}$$

The difference between $\chi\mathrm{PT}_3$ and $\chi\mathrm{PT}_\sigma$ is summarized in Fig. IV. See Appendix A for a discussion of power counting for $\chi\mathrm{PT}_\sigma$ loop expansions.

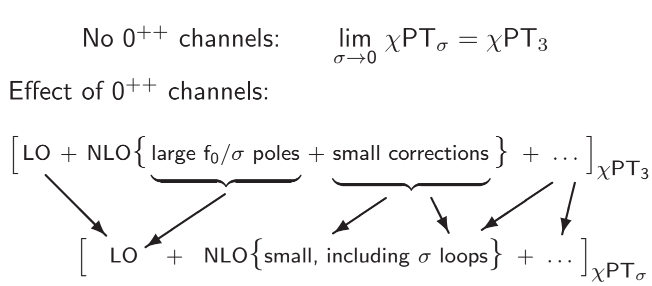

FIG. 7: Comparison of $\chi\mathrm{PT}_3$ and $\chi\mathrm{PT}_\sigma$. The f_0/σ pole terms responsible for the poor convergence of $\chi\mathrm{PT}_3$ are transferred to LO in $\chi\mathrm{PT}_\sigma$, where they do not upset convergence.

[12] Beyond lowest order, and in degenerate cases like the K_L–K_S mass difference, methods used to estimate corrections at the Z^0 peak [90] and the ρ resonance [91] may be necessary.

V. RESONANCE SATURATION IN χPT$_\sigma$

Conventional χPT$_3$ is often supplemented by a technique [92] in which the coefficients of $O(\partial^4) = O(m_K^4)$ terms are estimated by saturation with particles or resonances from the non-NG sector. This scheme can be readily adapted to χPT$_\sigma$, provided that the changed role of f_0/σ is understood.

Each non-NG particle or resonance of mass M_{res} gives rise to a pole factor which carries a linear combination $p = O(m_K)$ of the external momenta. The relevant coefficient is obtained from terms $\sim p^4/M_{\text{res}}^2$ in *heavy-particle* expansions of these pole factors

$$p^4\big/\big(p^2 - M_{\text{res}}^2\big) = p^2 + p^4/M_{\text{res}}^2 + \ldots \quad \text{for } M_{\text{res}} \gg p\,. \tag{69}$$

These expansions are nonchiral, i.e. they are not light-particle, small-momentum expansions of the type (16). Evidently this technique assumes a clear scale separation between the NG and non-NG sectors.

Where does the $f_0(500)$ resonance fit into this scheme? Having it contribute as a light particle in chiral expansions and a heavy particle in Eq. (69) would be double counting.

In χPT$_3$, the answer is that the $f_0(500)$ does not belong to the NG sector, so it is treated as a heavy resonance. The obvious lack of scale separation with the K, η NG bosons (Fig. 4) makes this proposal unworkable.

In χPT$_\sigma$, the problem disappears because f_0/σ is assigned to the NG sector. Its contributions are already taken into account in chiral expansions, so logically, it must be *excluded* from the saturation procedure of [92]. That is in line with the requirement that saturation be restricted to the *non-NG sector*. Scale separation of the NG and non-NG sectors works well for χPT$_\sigma$ (Fig. 4), so the heavy-particle conditions $M_{\text{res}} \gg p$ for $p = O(m_K)$ are satisfied.

In practice, χPT$_\sigma$ coefficients such as those in Eqs. (67) and (68) are not easily evaluated, because the analysis requires data for soft σ as well as soft π, K, η amplitudes.

VI. ELECTROMAGNETIC PROPERTIES OF MESONS

In χPT$_\sigma$, the electromagnetic interactions of NG bosons are of great interest because

- The amplitudes for $K_S \to \gamma\gamma$ and $\gamma\gamma \to \pi\pi$ can be used to analyse $K \to 2\pi$ (Sec. VII).
- The electromagnetic trace anomaly (8) and hence the Drell-Yan ratio can be estimated at the infrared fixed point $\alpha_s = \alpha_{\text{IR}}$.
- In $\gamma\gamma$ channels, meson loops can produce Li-Pagels singularities $\sim 1/m_{\pi,K,\sigma}^2$ and hence amplitudes which compete with σ-pole tree diagrams.

Photon interactions are introduced as in χPT$_3$, with the added requirement that the chiral singlet field σ is gauge invariant. So under local $U(1)$ transformations, we have

$$\sigma \to \sigma\,, \quad U \to e^{-i\lambda(x)Q} U e^{i\lambda(x)Q}\,, \tag{70}$$

where $Q = \frac{1}{3}\text{diag}(2, -1, -1)$ is the quark-charge matrix. Gauge invariance can be satisfied minimally by introducing a covariant derivative for U,

$$D_\mu U = \partial_\mu U + ieA_\mu[Q, U]\,, \tag{71}$$

where A_μ is the photon field. However this is not sufficient: it does not change the scaling properties of the effective Lagrangian, and so cannot produce an electromagnetic trace anomaly (8) proportional to $F_{\mu\nu}F^{\mu\nu}$.

The operator $F_{\mu\nu}F^{\mu\nu}$ has dimension 4, so we need an action which, when varied, produces a scale *invariant* result. This can happen only if the scaling property is *inhomogeneous*. The σ field has a scaling property (47) of that type, from which it is evident that the effective Lagrangian must contain a nonminimal term of the form

$$\mathcal{L}_{\sigma\gamma\gamma} = \tfrac{1}{4} g_{\sigma\gamma\gamma} \sigma F_{\mu\nu}F^{\mu\nu}\,. \tag{72}$$

This is the effective vertex first considered by Schwinger [93] in his study of the gauge invariance of fermion triangle diagrams.

Originally, the electromagnetic trace anomaly (8) was derived in the context of broken scale invariance (before QCD and asymptotic freedom), so the ultraviolet limit defining the Drell-Yan ratio R was nonperturbative. A comparison of Eqs. (8) and (72) in the tree approximation, or equivalently σ-pole dominance of θ_μ^μ (PCDC), led to the conclusion [37, 38] that the coupling of σ to $\gamma\gamma$ is proportional to R.

In the current context, there are two important modifications to this argument.

The first is to identify "R" correctly. In the physical region $0 < \alpha_s < \alpha_{\text{IR}}$, asymptotic freedom controls the ultraviolet limit and produces a perturbative answer

$$R_{\text{UV}} = \sum\{\text{quark charges}\}^2 = 2\,,\ N_f = N_c = 3 \tag{73}$$

for $N_f = 3$ light flavors and $N_c = 3$ colors. However, the hard gluonic operator G^2 in θ_μ^μ prevents PCDC from being used to relate low-energy amplitudes to asymptotically free quantities like R_{UV}. Instead, in the lowest order of χPT$_\sigma$, we use amplitudes defined at the infrared fixed point where the gluonic trace anomaly vanishes and so PCDC can be tested. At the infrared fixed point $\alpha_s = \alpha_{\text{IR}}$, there is no asymptotic freedom, so the UV limit of $e^+e^- \to$ hadrons produces a *nonperturbative* value R_{IR} which has to be determined theoretically. Thus we expect $g_{\sigma\gamma\gamma}$ to be related to R_{IR}.

The second modification is a surprise. In $\gamma\gamma$ channels, meson-loop integrals produce inverse Li-Pagels singularities $\sim M^{-1}$ in the chiral limit $M \sim 0$, where M is the

π, K, η (mass)^2 matrix (39). These infrared singularities are strong enough to allow $\pi^\pm, K^\pm$ one-loop diagrams to have the *same* chiral order as tree amplitudes containing the anomalous vertex in (72). This means that naive PCDC (σ-pole dominance) does not work when $\gamma\gamma$ channels are present; for example, the $\sigma \to \gamma\gamma$ coupling turns out to be proportional to $(R_{\rm IR} - 1/2)$, not $R_{\rm IR}$. Similar problems are not encountered for PCAC, partly because loop corrections to PCAC are limited by the negative parity of the corresponding Nambu-Goldstone bosons.

It becomes less surprising when the power-counting rule (7) for electromagnetic corrections to χPT expansions is considered.

A standard treatment of χPT [34, 35] is to require that the effective Lagrangian be invariant under *local* chiral $SU(N_f)_L \times SU(N_f)_R$ transformations. This requirement is satisfied minimally by replacing ordinary derivatives ∂_μ acting on U fields with covariant ones

$$D_\mu U = \partial_\mu U - \tfrac{i}{2}(v_\mu + a_\mu)U + \tfrac{i}{2}U(v_\mu - a_\mu)\,, \qquad (74)$$

where the gauge fields $v_\mu(x)$ and $a_\mu(x)$ transform inhomogeneously under the respective vector and axial-vector subgroups of $SU(N_f)_L \times SU(N_f)_R$. In order to match the chiral counting $\partial_\mu U = O(p)$ used by Weinberg [28] to study pure pion processes in χPT$_2$, Gasser and Leutwyler proposed the rule [34, 35]

$$a_\mu \sim v_\mu = O(p)\,. \qquad (75)$$

For electromagnetic processes, this requires the photon field A_μ obtained from

$$v_\mu = -2eQA_\mu \qquad \text{and} \qquad a_\mu = 0, \qquad (76)$$

to be counted as $O(p)$. As a result, one-loop meson amplitudes which couple (say) σ to any number of external photons are of the same chiral order, namely $O(p^4)$.

In χPT$_\sigma$, where the global symmetry group includes dilatations, chiral gauge invariance is not sufficient to determine the chiral order for nonminimal operators such as (72). In Appendix B, we generalize the Gasser-Leutwyler analysis to cover such cases. As a result:

1. Both Eq. (75) and the rule $A_\mu = O(p)$ remain valid.

2. The operator (72) gives rise to a $O(p^4)$ vertex amplitude of the same chiral order as one-loop meson graphs for $\sigma \to \gamma\gamma$.

3. In the presence of photons, χPT$_\sigma$ corrections to lowest-order tree and loop diagrams still converge: each additional loop is suppressed by a factor $\sim M \ln M$ or M.

In this Section, we consider lowest-order amendments to PCDC for the amplitude $\langle\gamma\gamma|\widetilde{\theta}^\mu_\mu|\text{vac}\rangle$.

Let $\gamma_i = \gamma(\epsilon_i, k_i)$ represent a photon with polarization ϵ_i and momentum k_i, and let $F(s)$ be the form factor defined by

$$\langle\gamma_1, \gamma_2|\widetilde{\theta}^\mu_\mu(0)|\text{vac}\rangle = (\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2 \epsilon_2\cdot k_1)F(s)\,. \quad (77)$$

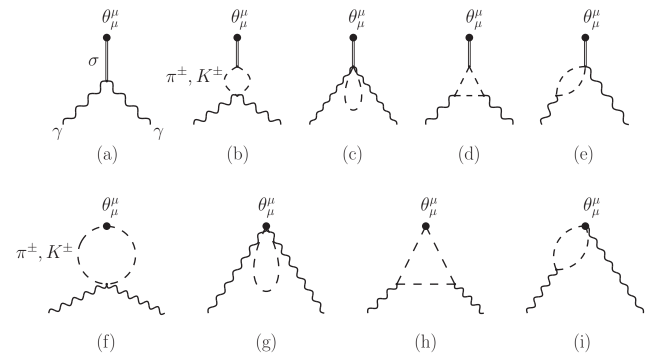

FIG. 8: Lowest order contributions to $\langle\gamma_1, \gamma_2|\widetilde{\theta}^\mu_\mu(0)|\text{vac}\rangle$ in χPT$_\sigma$. Diagram (a) represents the contact term proportional to $g_{\sigma\gamma\gamma}$, while diagrams (d), (e), (h), and (i) are each accompanied by an additional crossed amplitude (not shown). Similar loop diagrams have been considered in χPT$_3$ for $K_S \to \gamma\gamma$ [33], $K_L \to \pi^0\gamma\gamma$ [53], and $\gamma\gamma \to \pi^0\pi^0$ [55, 56].

The electromagnetic trace anomaly concerns the value of this form factor at $s = 0$:

$$F(0) = -\tfrac{1}{3}\pi\alpha \int d^4x\, d^4y\, x\cdot y\, T\langle J^\beta(x) J_\beta(0)\theta^\mu_\mu(y)\rangle_{\rm vac}\,. \qquad (78)$$

At the fixed point $\alpha_s = \alpha_{\rm IR}$, we have a theory of broken scale invariance, so the conditions of the derivations in [37, 38] are satisfied. The leading short-distance behavior of both $\langle J_\alpha J_\beta \theta_{\mu\nu}\rangle_{\rm vac}$ and $\langle J_\alpha J_\beta\rangle_{\rm vac}$ is conformal, with no anomalous dimensions because J_α and $\theta_{\mu\nu}$ are conserved, and the soft $d < 4$ trace θ^μ_μ ensures convergence of Eq. (78) at $x \sim y \sim 0$. Therefore, we can write down an exact anomalous Ward identity[13]

$$F(0) = \frac{2R_{\rm IR}\alpha}{3\pi}\,,\ \alpha_s = \alpha_{\rm IR}\,. \qquad (79)$$

The calculation of the form factor $F(s)$ in χPT$_\sigma$ involves two classes of diagrams (Fig. 8):

1. Dilaton pole diagrams (a-e) which produce a factorized amplitude
$$F_1(s) = \mathcal{A}_{\sigma\gamma\gamma}\frac{i}{s - m_\sigma^2}(-m_\sigma^2 F_\sigma)\,. \qquad (80)$$
Here $\mathcal{A}_{\sigma\gamma\gamma}$ includes a contact term $-ig_{\sigma\gamma\gamma}$ from diagram (a) and contributions from one-loop diagrams (b-e) with internal $\pi^\pm, K^\pm$ lines.

2. A one-loop amplitude $F_2(s)$ from diagrams (f-i) with internal $\pi^\pm, K^\pm$ lines coupled to the vacuum via θ^μ_μ.

[13] There is similar result for $0 < \alpha_s < \alpha_{\rm IR}$ which involves $R_{\rm UV}$ but has no practical use. See Appendix C.

The $\sigma \to \gamma\gamma$ amplitude in Eq. (80) can be written

$$\mathcal{A}_{\sigma\gamma\gamma} = -ig_{\sigma\gamma\gamma} + \frac{i\alpha}{\pi F_\sigma}\mathcal{C}\sum_{\phi=\pi,K} m_\phi^2\Big(\frac{1+2I_\phi}{s}\Big) \tag{81}$$

where the label $\phi = \pi^\pm$ or $K^\pm$ refers to the meson propagating around the loop in diagrams (b-e). In Eq. (81), the constant $\mathcal{C}$ is a combination of low energy coefficients

$$\mathcal{C} = 1 - \gamma_m - (1-c_1)\beta' \tag{82}$$

and I_ϕ is the relevant Feynman-parametric integral

$$I_\phi = m_\phi^2 \iint_0^1 dz_1 dz_2\, \theta(1-z_1-z_2)\big/\big(z_1 z_2 s - m_\phi^2\big) \tag{83}$$

for on-shell photons $k_1^2 = k_2^2 = 0$. The constant $\mathcal{C}$ and integral I_ϕ also appear in the result for diagrams (f-i):

$$F_2(s) = \frac{\alpha}{\pi}(\mathcal{C}-2)\sum_{\phi=\pi,K} m_\phi^2\Big(\frac{1+2I_\phi}{s}\Big)\,. \tag{84}$$

The final step is to compare the answer for

$$F(s) = F_1(s) + F_2(s) \tag{85}$$

with the $s = 0$ constraint (79). For that, we need the Taylor expansion

$$1 + 2I_\phi = -\frac{s}{12m_\phi^2} + O(s^2)\,. \tag{86}$$

Summing the $\pi^\pm$ and $K^\pm$ contributions, we have

$$\sum_{\phi=\pi,K} m_\phi^2\Big(\frac{1+2I_\phi}{s}\Big) = -\frac{1}{6} + O(s)\,, \tag{87}$$

and so find that the terms involving $\mathcal{C}$ cancel:

$$F(s) = g_{\sigma\gamma\gamma}F_\sigma + \alpha/3\pi + O(s)\,. \tag{88}$$

Comparison with Eq. (79) yields the desired relation[14]

$$g_{\sigma\gamma\gamma} = \frac{2\alpha}{3\pi F_\sigma}\Big(R_{\mathrm{IR}} - \tfrac{1}{2}\Big)\,. \tag{89}$$

Evidently, the one-loop diagrams which produce the term $-\frac{1}{2}$ relative to R_{IR} have the same chiral order as the tree diagram involving $g_{\sigma\gamma\gamma}$. This is an explicit demonstration of the way PCDC is modified by the inverse Li-Pagels singularities noted above for $\gamma\gamma$ channels.

An estimate for R_{IR} from Eq. (89) is not straightforward because dispersive analyses of reactions such as $\gamma\gamma \to \pi\pi$ yield residues at the f_0/σ pole proportional to the full amplitude $\mathcal{A}_{\sigma\gamma\gamma}(s = m_\sigma^2)$ of Eq. (81), not $g_{\sigma\gamma\gamma}$. Currently, we have no independent data about the constant $\mathcal{C}$, apart from the weak constraint (56) for $(1-c_1)\beta'$ and the inequality

$$-1 \leqslant 1 - \gamma_m < 2 \tag{90}$$

from Eq. (35). We will argue below that numerically, these corrections are likely to be small compared with the electromagnetic trace anomaly. First, let us review what is known about $\gamma\gamma \to \pi\pi$ from dispersion theory.

The residue of the $f_0(500)$ pole was first extracted from the Crystal Ball data [58] by Pennington [63] and subsequently refined in several analyses [94–97]. We use a recent determination [97] of the radiative width

$$\Gamma_{\sigma\gamma\gamma} = 2.0 \pm 0.2 \text{ keV} \tag{91}$$

based on fits to data [98] of pion polarizabilities.[15]

In lowest order $\chi\mathrm{PT}_\sigma$, the relevant diagrams for the process $\sigma \to \gamma\gamma$ are those shown in (a-e) of Fig. 8, but with σ treated as an asymptotic state. The narrow width approximation is valid in lowest order $\chi\mathrm{PT}_\sigma$, so the magnitude of the full amplitude $\mathcal{A}_{\sigma\gamma\gamma}$ at $s = m_\sigma^2$ is determined by

$$\Gamma_{\sigma\gamma\gamma} = \frac{m_\sigma^3}{64\pi}|\mathcal{A}_{\sigma\gamma\gamma}|^2\,. \tag{92}$$

Comparison with (91) then gives

$$|\mathcal{A}_{\sigma\gamma\gamma}| = 0.068 \pm 0.006 \text{ GeV}^{-1} \tag{93}$$

where the uncertainties have been added in quadrature.

The presence of lowest order meson loops in $\gamma\gamma$ channels implies that numerical results for the contact term depend on how the scalar field is defined.[14] Consequently, care must be exercised when comparing our value with those found using $\chi\mathrm{PT}_3$ or dispersion theory — definitions of "the contact $f_0\gamma\gamma$ coupling" are not necessarily equivalent. For example, the small values for these couplings reported in dispersive analyses [103, 104] could well be consistent with each other and with our result for the coupling $\mathcal{L}_{\sigma\gamma\gamma}$ of Eq. (72).

In $\chi\mathrm{PT}_\sigma$ we find that for N_c large, it is the contact term which is the dominant contribution to $\mathcal{A}_{\sigma\gamma\gamma}$. This is because, relative to the single-quark loop diagrams associated with $R_{\mathrm{IR}} = O(N_c)$, terms from $\pi^\pm, K^\pm$ loop graphs involve an additional quark loop and so are suppressed by a factor $1/N_c$. We therefore have

$$g_{\sigma\gamma\gamma} = O\big(\sqrt{N_c}\big) \quad \text{and} \quad \mathcal{C} = O(1) \tag{94}$$

[14] The answer is simple because we chose a σ field with the scaling property (47). Constants like $\mathcal{C}$ can appear if other definitions of σ are used.

[15] We do not use the alternative estimate [97] $\Gamma_{\sigma\gamma\gamma} = 1.7 \pm 0.4$ keV because it depends on scalar meson resonance saturation for low energy constants of $\chi\mathrm{PT}_2$ expansions [99, 100] and also (tracing back via App. D.2.2 of [101] to [102]) $\chi\mathrm{PT}_3$ expansions. As noted in Sec. IV below Eq. (69), that places f_0 in the non-NG sector. It would be inconsistent for us to combine that with $\chi\mathrm{PT}_\sigma$.

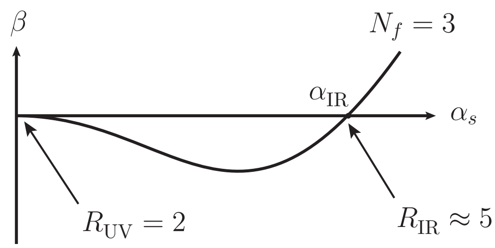

FIG. 9: Drell-Yan ratios $R_{\rm UV}$ and $R_{\rm IR}$ associated with the proposed β/ψ function. For $e^+e^- \to$ hadrons at high energies with $0 < \alpha_s < \alpha_{\rm IR}$, the strong coupling α_s runs to zero and the result $R_{\rm UV}$ is perturbative (asymptotic freedom). However if α_s is at $\alpha_{\rm IR}$, it cannot run, so we get a nonperturbative result $R_{\rm IR}$ associated with *short*-distance scaling at the infrared fixed point.

in the large-N_c limit and conclude[16]

$$\mathcal{A}_{\sigma\gamma\gamma} = -ig_{\sigma\gamma\gamma} + O\big(1/\sqrt{N_c}\big)\,. \tag{95}$$

From Eq. (93) and within the large uncertainty due to that in F_σ, we estimate

$$R_{\rm IR} \approx 5\,. \tag{96}$$

This result is a feature of the nonperturbative theory at $\alpha_{\rm IR}$ (Fig. 9), so it has *nothing* to do with asymptotic freedom or the free-field formula (73).

VII. WEAK INTERACTIONS OF MESONS

The most important feature of χPT$_\sigma$ is that it explains the empirical $\Delta I = 1/2$ rule for nonleptonic kaon decays such as $K \to \pi\pi$.

Problems explaining the data for nonleptonic kaon and hyperon decays were first recognised sixty years ago [105]. They became acute with the advent of three-flavor chiral perturbation theory. For χPT$_3$ applied to kaon decays, the dilemma is:

1. A fit to data in lowest nontrivial order, i.e. for $O(p^2)$ amplitudes $\mathcal{A}_{\rm LO}$, would require the ratio of **8** to **27** couplings $|g_8/g_{27}|$ to be $\simeq 22$, much larger than any of the reasonable estimates in the range (6).

2. The main alternative is to accept Eq. (6) and argue that the dominant contribution comes from a NLO $O(p^4)$ term produced by strong final-state interactions in the 0^{++} channel, e.g. via a non-NG scalar boson S [42–45] for which the pole diagram in Fig. 2 is $O(p^4/m_S^2)$, with $m_S \neq 0$. Then the χPT$_3$ expansion diverges uncontrollably,[17]

$$\big|{\rm NLO}/{\rm LO}\big|_{\chi{\rm PT}_3} \simeq 22 \tag{97}$$

contradicting the premise that χPT$_3$ is applicable.

Let us review option 1 in more detail. In the lowest order[18] of standard χPT$_3$, the effective weak Lagrangian

$$\mathcal{L}_{\rm weak}\big|_{\sigma=0} = g_8 Q_8 + g_{27} Q_{27} + Q_{mw} + \text{h.c.} \tag{98}$$

contains an octet operator [107]

$$Q_8 = \mathcal{J}_{13}\mathcal{J}_{21} - \mathcal{J}_{23}\mathcal{J}_{11}\ , \quad \mathcal{J}_{ij} = (U\partial_\mu U^\dagger)_{ij} \tag{99}$$

the U-spin triplet component [31, 108] of a **27** operator

$$Q_{27} = \mathcal{J}_{13}\mathcal{J}_{21} + \tfrac{3}{2}\mathcal{J}_{23}\mathcal{J}_{11} \tag{100}$$

and a weak mass operator [109]

$$Q_{mw} = {\rm Tr}(\lambda_6 - i\lambda_7)\big(g_M M U^\dagger + \bar{g}_M U M^\dagger\big)\,. \tag{101}$$

Although Q_{mw} has isospin 1/2, it cannot be used to solve the $\Delta I = 1/2$ puzzle if dilatons are absent. When Q_{mw} is combined with the strong mass term $\mathcal{L}_{\rm mass}|_{\sigma=0}$, it can be removed by a chiral rotation

$$U \to \widetilde{U} = RUL^\dagger \tag{102}$$

which aligns the vacuum such that

$$\langle\widetilde{U}\rangle_{\rm vac} = I \quad \text{and} \quad M = \text{real diagonal}. \tag{103}$$

Therefore [108] Q_{mw} has no effect on χPT$_3$ low-energy theorems relating $K \to \pi\pi$ and $K \to \pi$ on shell, and so the conclusion that $|g_8/g_{27}|$ is unreasonably large (≈ 22) cannot be avoided.

In χPT$_\sigma$, the outcome is entirely different. First, we adjust the operator dimensions of Q_8, Q_{27}, and Q_{mw} by powers of e^{σ/F_σ}

$$\begin{aligned}\mathcal{L}_{\rm weak} = Q_8 \sum_n g_{8n} e^{(2-\gamma_{8n})\sigma/F_\sigma} + g_{27}Q_{27}e^{(2-\gamma_{27})\sigma/F_\sigma} \\ + Q_{mw}e^{(3-\gamma_{mw})\sigma/F_\sigma} + \text{h.c.}\,,\end{aligned} \tag{104}$$

as in Eqs. (49) and (68) for the strong interactions, with octet quark-gluon operators allowed to have differing dimensions at $\alpha_{\rm IR}$. The key point is that the weak mass operator's dimension $(3-\gamma_{mw})$ bears no relation to the dimension $(3-\gamma_m)$ of $\mathcal{L}_{\rm mass}$, so the σ dependence of

[16] This approximation is *not* required in our analysis of $K_S \to \pi\pi$ in Sec. VII. Indeed $g_{\sigma\gamma\gamma}$ does not appear anywhere. The key ingredient is the phenomenological estimate (93) for the complete amplitude $\mathcal{A}_{\sigma\gamma\gamma}$.

[17] The factor 22 is 70 times larger than the limit ~ 0.3 prescribed by Eq. (13) for an acceptable fit.

[18] Our aim is to solve the $\Delta I = 1/2$ puzzle *without* using NLO terms. Weak NLO terms in χPT$_3$ [106], except those depending on f_0/σ (Sec. V), become weak NLO χPT$_\sigma$ terms when multiplied (as in Eq. (68)) by suitable powers of e^{σ/F_σ}. We expect these to produce small corrections to our result.

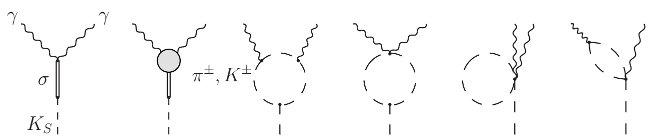

FIG. 10: Lowest order diagrams for $K_S \to \gamma\gamma$ in $\chi\mathrm{PT}_\sigma$, including finite loop graphs [33]. The grey vertex contains $\pi^\pm$, $K^\pm$ loops as in the four $\chi\mathrm{PT}_3$ diagrams to the right. An analogous set of diagrams contributes to $\gamma\gamma \to \pi^0\pi^0$.

$Q_{mw}e^{(3-\gamma_{mw})/F_\sigma}$ cannot be eliminated by a chiral rotation. Instead, after aligning the vacuum, we find

$$\mathcal{L}^{\mathrm{align}}_{\mathrm{weak}} = \widetilde{Q}_8 \sum_n g_{8n} e^{(2-\gamma_{8n})\sigma/F_\sigma} + g_{27}\widetilde{Q}_{27} e^{(2-\gamma_{27})\sigma/F_\sigma} + \widetilde{Q}_{mw}\big\{e^{(3-\gamma_{mw})\sigma/F_\sigma} - e^{(3-\gamma_m)\sigma/F_\sigma}\big\} + \mathrm{h.c.}\,, \qquad (105)$$

where the tilde indicates that the **8** and **27** operators are now functions of the rotated field $\widetilde{U}$. As a result, there is a residual interaction $\mathcal{L}_{K_S\sigma} = g_{K_S\sigma}K_S\sigma$ which mixes K_S and σ in *lowest* $O(p^2)$ order

$$g_{K_S\sigma} = (\gamma_m - \gamma_{mw})\mathrm{Re}\{(2m_K^2 - m_\pi^2)\bar{g}_M - m_\pi^2 g_M\}F_\pi/2F_\sigma \qquad (106)$$

and produces the $\Delta I = 1/2$ amplitude $A_{\sigma\text{-pole}}$ of Fig. 2.

At this point, we could simply choose $g_{K_S\sigma}$ to fit the rate for $K_S \to \pi\pi$, knowing that inserting the full $K_S \to \pi\pi$ amplitude into the standard loop calculation for $K_S \to \gamma\gamma$ [33] would give agreement with experiment. That would leave unclear what version of chiral perturbation theory in Fig. 4 is being used to analyse $K_S \to \gamma\gamma$.

So instead, we first apply $\chi\mathrm{PT}_\sigma$ to $K_S \to \gamma\gamma$ and $\gamma\gamma \to \pi\pi$ in order to determine $g_{K_S\sigma}$, and then show that this gives a result for $K_S \to \pi\pi$ which agrees with experiment.

The scalar part $\mathcal{A}_{K\gamma\gamma}$ of the $K_S \to \gamma\gamma$ amplitude

$$\mathcal{A}_{\mu\nu} = (g_{\mu\nu}k_1\cdot k_2 - k_{2\mu}k_{1\nu})\mathcal{A}_{K\gamma\gamma} \qquad (107)$$

receives three contributions at lowest order (Fig. 10)

$$\mathcal{A}_{K\gamma\gamma} = \mathcal{A}^{\mathrm{tree}}_\sigma + \mathcal{A}^{\mathrm{loop}}_\sigma + \mathcal{A}^{\mathrm{loop}}_{\pi,K}\,. \qquad (108)$$

The explicit expressions are

$$\mathcal{A}^{\mathrm{tree}}_\sigma + \mathcal{A}^{\mathrm{loop}}_\sigma = -2g_{K_S\sigma}\mathcal{A}_{\sigma\gamma\gamma}\big/\big(m_K^2 - m_\sigma^2\big)\,,$$
$$\mathcal{A}^{\mathrm{loop}}_{\pi,K} = -2\frac{\alpha}{\pi F_\pi^3}\big(g_8 + g_{27}\big)\sum_{\phi=\pi,K}(m_K^2 - m_\phi^2)\Big(\frac{1+2I_\phi}{m_K^2}\Big)\,, \qquad (109)$$

where the magnitude of $\mathcal{A}_{\sigma\gamma\gamma}$ is determined from Eq. (93) and I_ϕ is the integral given by Eq. (83). If we neglect the g_8 and g_{27} terms, we find

$$|g_{K_S\sigma}| \approx 4.4\times 10^3\,\mathrm{keV}^2 \qquad (110)$$

to a precision $\lesssim 30\%$ expected for a three-flavor chiral expansion.

Now consider $K_S \to \pi\pi$ (Fig. 2). Eq. (110) and data for the f_0 width (Eq. (57)) imply that the σ-pole diagram contributes (very roughly, given[12] σ/f_0's width and near degeneracy with K)

$$|A_{\sigma\text{-pole}}| = \left|\frac{-ig_{K_S\sigma}g_{\sigma\pi\pi}}{m_K^2 - m_\sigma^2}\right| \approx 0.34\,\mathrm{keV} \qquad (111)$$

to the full $I = 0$ amplitude[19]

$$A_0 = \frac{\sqrt{3}}{F_\pi^3}\big(g_8 + \tfrac{1}{6}g_{27}\big)\big(m_K^2 - m_\pi^2\big) + A_{\sigma\text{-pole}}\,. \qquad (112)$$

If the $g_{8,27}$ contributions are again neglected,

$$|A_0| \simeq |A_{\sigma\text{-pole}}| \qquad (113)$$

we see that Eq. (111) accounts for the large magnitude of A_0 [3]:

$$|A_0|_{\mathrm{expt.}} = 0.33\,\mathrm{keV}\,. \qquad (114)$$

We conclude that the observed ratio $|A_0/A_2| \simeq 22$ is mostly due to the dilaton-pole diagram of Fig. 2, that $g_8 = \sum_n g_{8n}$ and g_{27} have roughly similar magnitudes as simple calculations [29–32] indicate, and that only g_{27} can be fixed precisely (from $K^+ \to \pi^+\pi^0$).

Consequently, the lowest $O(p^2)$ order of $\chi\mathrm{PT}_\sigma$ solves the $\Delta I = 1/2$ problem for kaon decays. The chiral low-energy theorems which relate the on-shell[20] $K \to 2\pi$ and $K \to \pi$ amplitudes have extra terms due to σ poles, but the no-tadpoles theorem [108] is still valid:

$$\langle K|\mathcal{H}_{\mathrm{weak}}|\mathrm{vac}\rangle = O\big(m_s^2 - m_d^2\big)\,,\ K \text{ on shell.} \qquad (115)$$

VIII. REMARKS

Why must the 0^{++} particle be a dilaton in order to explain the $\Delta I = 1/2$ puzzle for K decays? Because the property $m_\sigma \to 0$ in the chiral-scale limit (2) is essential. As is evident from Eq. (97), assuming scalar dominance by a non-NG particle contradicts the basic premise of chiral theory that at low energies, the NG sector dominates the non-NG sector. That is why none of the authors proposing scalar dominance by a non-NG particle since 1980 [42] claimed to have solved the puzzle or persuaded others to stop working on other proposals, such

---

[19] Our convention for the $K \to \pi\pi$ isospin amplitudes is that given in [110].

[20] Ref. [108] followed standard practice in current algebra. It related the amplitude for $K \to \pi\pi$, where $\mathcal{H}_{\mathrm{weak}}$ carries zero 4-momentum, to $K \to \pi$ for *on-shell* kaons and pions, where the relevant operator $[F_5, \mathcal{H}_{\mathrm{weak}}]$ obviously carries nonzero 4-momentum. Ref. [108] is often misquoted by authors who implicitly set the momentum transfer for $K \to \pi$ equal to zero. In Eq. (115), $|\mathrm{vac}\rangle$ refers to the unique state with translation invariance, so $\mathcal{H}_{\mathrm{weak}}$ carries momenta whose square is m_K^2.

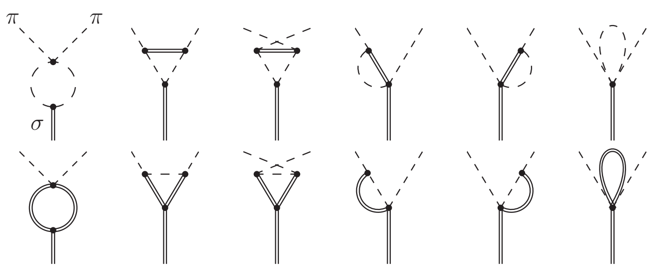

FIG. 11: Next to lowest order (NLO) diagrams which contribute to the resonance structure of f_0/σ in $\chi\mathrm{PT}_\sigma$. Ultraviolet divergences arising from the loops are absorbed (in the usual way) by counterterms in the NLO Lagrangian.

as penguin diagrams [111], the large-N_c limit [112, 113], or QCD sum rules [114].

Our resolution of the $\Delta I = 1/2$ puzzle is distinguished by not being *ad hoc*. It is part of a wider program to obtain numerically convergent three-flavor chiral expansions for amplitudes involving 0^{++} channels, i.e. where $\chi\mathrm{PT}_3$ clearly fails (Sec. II). So far, we can say only that lowest order $\chi\mathrm{PT}_\sigma$ appears to be a good approximation. More stringent tests of convergence have yet to be developed because loop corrections involve couplings like $\sigma\sigma\pi\pi$ for which we lack data. An important example is the shape of the σ resonance at NLO (Fig. 11), where the higher order corrections to (62) have yet to be determined. This will require explicit calculations which include numerical fits for the $\sigma\sigma\sigma, \sigma\sigma\pi\pi, \ldots$ couplings.

Another test could be to invent a unitarization procedure for $\chi\mathrm{PT}_\sigma$ and check whether (unlike $\chi\mathrm{PT}_3$) it produces *small* corrections to lowest order results.

The basis of our work on approximate scale and chiral $SU(3)_L \times SU(3)_R$ symmetry in QCD should be carefully distinguished from what is postulated in analyses of walking gauge theories. As noted by Del Debbio [15], in such theories, "the infrared fixed point . . . describes the physical properties of theories which are scale invariant at large distances, where the field correlators have power-like behaviours characterized by the anomalous dimensions of the fields." That means that there is no mass gap at the fixed point: scale condensates are assumed to be absent.

Our view of physics at the infrared fixed point is quite different. The Hamiltonian becomes scale invariant for massless u, d, s quarks, but the vacuum is *not* scale invariant because of the condensate $\langle \bar{q}q \rangle_{\mathrm{vac}} \neq 0$. It sets the scale of the mass gap for hadrons $\rho, \omega, N, \eta', \ldots$ in the non-NG sector (Sec. II below Eq. (28)). For example, at the infrared fixed point in Fig. 9, $e^+e^- \to$ hadrons at low or intermediate energies has thresholds and resonances similar to QCD at similar energies. Scaling behaviour sets in only at *high* energies.[3]

A result of this fundamental difference is that our hypothesis of an infrared fixed point for $N_f = 3$ is not tested by lattice investigations done in the context of walking gauge theories. Those investigations are based on criteria like Miransky scaling [115] which assume that a theory cannot have an infrared fixed point if it does not display the behavior described above in the quote from Del Debbio.

More generally, our view is that theoretical evidence for or against our proposal in Fig. 1 is inconclusive. Various definitions of the QCD running coupling can be be readily compared in low orders of perturbation theory, but it is not at all clear which definitions are physically equivalent beyond that. The key nonperturbative requirements for a running coupling are that its dependence on the magnitude of a space-like momentum variable be monotonic and analytic. Gell-Mann and Low [116] achieved this for QED, but these properties are hard to establish for QCD running couplings. A lack of equivalence of these definitions may explain why differing results for infrared fixed points are obtained.

Unfortunately, our analysis does not explain the failure of chiral theory to account for nonleptonic $|\Delta S| = 1$ hyperon decays. We have shown that octet dominance is not necessary for K-decays, but that makes no difference for hyperon decays: the Pati-Woo $\Delta I = 1/2$ mechanism [117] forbids all contributions from **27** operators.

Acknowledgments

We thank Ross Young, Peter Minkowski, Martin Hoferichter, and Gilberto Colangelo for valuable discussions at various stages of this work. LCT thanks Mary K. Gaillard and her colleagues for their kind hospitality at the Berkeley Center for Theoretical Physics, where part of this work was completed. LCT is supported in part by the Australian-American Fulbright Commission, the Australian Research Council, the U.S. Department of Energy under Contract DE-AC02-05CH11231, the National Science Foundation under grant PHY-0457315, and the Federal Commission for Scholarships for Foreign Students (FCS). The Albert Einstein Center for Fundamental Physics at the University of Bern is supported by the "Innovations- und Kooperationsprojekt C-13" of the "Schweizerische Universitätskonferenz SUK/CRUS".

Note Added. A similar chiral Lagrangian with a technicolor "dilaton" has just been published by Matsuzaki and Yamawaki [118]. They acknowledge our prior work [4], but say that they do not believe it to be valid for hadronic physics. The basis for this assertion is the claim (footnote 8 of [16]) that "light" dilatons are forbidden by the one-loop formula $-(6\pi)^{-1}(33 - 2N_f)\alpha_s^2$ for the QCD β function. The problems with this are that (a) the relevant limit is infrared, not ultraviolet, and (b) for α_s large, the one-loop formula violates the analyticity bound [119] $\beta \gtrsim -\alpha_s \ln \alpha_s$.

Appendix A: Chiral Order in $\chi\mathrm{PT}_\sigma$

In 1979, Weinberg [28] found that successive terms in the $\chi\mathrm{PT}_2$ expansion of amplitudes with pionic external and internal lines and external momenta $p \sim m_\pi$ obey the "power counting" rule that each additional loop produces a factor $\sim p^2$ or, if there is a Li-Pagels singularity [40, 41], $p^2 \ln p$. With essentially no change in the analysis, this rule can be extended to $\chi\mathrm{PT}_3$ for pure π, K, η amplitudes with $p \sim m_\pi, m_K, m_\eta$. Here we extend Weinberg's rule to $\chi\mathrm{PT}_\sigma$ for amplitudes with internal and external lines restricted to the corresponding NG sector π, K, η, σ (Fig. 4).

This extension is not entirely straightforward because $\chi\mathrm{PT}_\sigma$ is really the special case

$$\alpha_s - \alpha_{\mathrm{IR}} = O(M \text{ or } \partial\partial) \quad \text{for} \quad \partial\partial = O(M) \tag{A1}$$

of a double expansion[10] in the quark mass matrix M about zero and the running coupling α_s about α_{IR}. In higher orders, critical exponents such as $\beta'(\alpha_{\mathrm{IR}})$ and $\gamma_m(\alpha_{\mathrm{IR}})$ become leading terms of expansions in M. For example, when $\gamma_{G^2}(\alpha_s)$ in Eq. (43) is expanded about the fixed point, the dimension (44) of the gluonic anomaly is corrected as follows:

$$d_{\mathrm{anom}} \to 4 + \beta'(\alpha_{\mathrm{IR}}) + \big(\alpha_s - \alpha_{\mathrm{IR}}\big)\gamma'_{G^2}(\alpha_{\mathrm{IR}}) + \ldots \tag{A2}$$

Then terms in the $\chi\mathrm{PT}_\sigma$ Lagrangian will include corrections of the form

$$e^{\beta'\sigma/F_\sigma} \to e^{\beta'\sigma/F_\sigma}\big\{1 + \mathcal{O}\big\} \tag{A3}$$

where $\mathcal{O}$ accounts for the effects of powers of the QCD factor $\alpha_s - \alpha_{\mathrm{IR}}$. In the effective theory, this factor may correspond to either an explicit M factor or two derivatives $\partial \ldots \partial$ not necessarily acting on the same field. A power $(\alpha_s - \alpha_{\mathrm{IR}})^p$ will then produce a superposition of terms

$$\sim \{2k\ \partial \text{ operators on up to } 2k \text{ fields}\} M^{p-k}\,. \tag{A4}$$

Therefore $\mathcal{O}$ is in general an operator depending on powers of σ, M and $\partial\partial$.

So in a higher chiral order, terms in the effective Lagrangian may involve σ-dependent factors

$$\sigma^{\text{integer } > \ 0} \exp\big(\{\text{constant}\}\sigma/F_\sigma\big) \tag{A5}$$

which do *not* scale homogenously under the transformations (47). Indeed, terms of that type appear as renormalization counterterms for $\chi\mathrm{PT}_\sigma$ loop expansions. The proliferation of low-energy coupling constants due to inhomogenous scaling, with constraints between them possible but not obvious, makes the phenomenology of higher-order $\chi\mathrm{PT}_\sigma$ challenging.

Fortunately, these complications do not impede the extension of Weinberg's rule to $\chi\mathrm{PT}_\sigma$. Our approach resembles Sec. 3.4.9 of the review [72].

Let ϕ refer to the spin 0^- octet π, K, η. In momentum space, a general vertex involving σ and ϕ fields produces terms

$$\sim p_v^d m_\phi^{2k} m_\sigma^{2k'}\ , \quad \text{integers } d, k, k' \tag{A6}$$

where p_v refers to components of the various vertex momenta and the product p_v^d has degree d when all p_v are scaled to tp_v. The aim is to determine the behavior of Feynman diagrams under the rescaling

$$p_e \to tp_e\,, \qquad m_\phi^2 \to t^2 m_\phi^2\,, \qquad m_\sigma^2 \to t^2 m_\sigma^2 \tag{A7}$$

of all external momenta p_e and masses $m_{\phi,\sigma}$ of the NG bosons ϕ and σ. Note that the dilaton mass m_σ scales in the same way as m_ϕ, in keeping with the discussion[10] of Eqs. (25), (45) and (A1).

Let $\mathcal{A}(p_e, m_\phi, m_\sigma)$ be a connected amplitude given by a diagram with I_ϕ internal ϕ lines, I_σ internal σ lines, and $N_{dkk'}$ vertices of the form (A6). External lines are amputated (e.g. placed on shell) and the factor $\delta^4(\sum p_e)$ for momentum conservation is omitted. Apart from Li-Pagels logarithms [40, 41] produced by loop integrals, each internal NG boson line

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_{\phi,\sigma}^2 + i\epsilon} \tag{A8}$$

contributes a factor t^2 under (A7), so $\mathcal{A}$ scales with a chiral dimension or order given by

$$D = 4 + 2I_\phi + 2I_\sigma + \sum_{d,m,m'} N_{dmm'}(d + 2m + 2m' - 4)\,. \tag{A9}$$

The number of independent loops N_ℓ for a graph is related to the total number of vertices $V = \sum_{dmm'} N_{dmm'}$ by the geometric identity

$$N_\ell = I_\phi + I_\sigma - V + 1\,. \tag{A10}$$

Substituting this identity into (A9) gives a result

$$D = 2 + \sum_{d,m,m'} N_{dmm'}(d + 2m + 2m' - 2) + 2N_\ell\,, \tag{A11}$$

similar to that obtained originally [28] by Weinberg.

The feature of Eq. (A11) worth emphasizing is that the loop number N_ℓ places a lower bound on D. That is because the vertex contribution (A6) must have chiral dimension $\geqslant 2$, i.e.

$$d + 2m + 2m' - 2 \geqslant 0 \tag{A12}$$

and so (A11) implies the general inequality

$$D \geqslant 2 + 2N_\ell\,. \tag{A13}$$

The case $D = 2$ includes and is limited to tree graphs produced in lowest order, i.e. by vertices (A6) with chiral dimension 2.

Given that the Li-Pagels infrared singularity for N_ℓ-loop diagrams is $O(\ln^\ell t)$ at most, we see that each new loop is suppressed by a factor of at most $t^2 \ln t$ for small t, as in $\chi\mathrm{PT}_2$ and $\chi\mathrm{PT}_3$. The extra logarithm is too weak to allow a given loop diagram to compete with diagrams with a smaller number of loops. In particular, both σ-pole dominance (PCDC) and ϕ-pole dominance (PCAC) are valid for pure NG processes in lowest order. This is consistent with the discussion of the σ width in Sec. IV.

A further result is that higher-order versions of the constraint (45) on $\mathcal{L}_{\mathrm{anom}}$ are not needed. Once imposed, it can be maintained to any order.

Apart from remarks about the σNN coupling in Sec. II, the analysis in this paper is limited to the NG sector. Chiral power counting in the presence of baryons and other non-NG particles is a nontrivial matter even for ordinary χPT [72].

Appendix B: External Currents and Wilson Operators in χPT

This Appendix concerns the effect of operators such as the lowest order $(\mathbf{8},\mathbf{1})$ and $(\mathbf{1},\mathbf{8})$ currents

$$\mathcal{F}_\mu = F_\pi^2 e^{2\sigma/F_\sigma} U i\partial_\mu U^\dagger \quad \text{and} \quad \mathcal{F}_\mu^\dagger = F_\pi^2 e^{2\sigma/F_\sigma} U^\dagger i\partial_\mu U \tag{B1}$$

on chiral power counting in the NG sector. This arises from the discussion in Sects. I and VI of the Gasser-Leutwyler rule (7) and the failure of naive σ-pole dominance (PCDC) for $\sigma \to \gamma\gamma$, where loop diagrams with inverse-power Li-Pagels singularities compete with the tree diagram. These singular powers are counted automatically if Appendix A is extended to include vertices due to external operators carrying low momenta.

Vertices of the currents (B1) have chiral order 1 because of the presence of a single derivative $\partial = O(p)$. At first sight, counting a single power for the corresponding current sources (7) and (75) seems obvious. For (say) a photon insertion in an internal propagator, we have

$$O\big(1/p^2\big) \longrightarrow O\big(A \times p/p^4\big) \tag{B2}$$

with no change in loop number, so the chiral order for amplitudes with photons can be matched to those without by choosing $A_\mu \sim p$.

What is less obvious is the idea that these rules remain valid for sources of currents in QCD itself, where they enter *linearly* in the action, e.g.

$$S_{\mathrm{QCD}} \longrightarrow S_{\mathrm{QCD}} + \int d^4x A_\mu \bar{q}\gamma^\mu Q q \tag{B3}$$

but give rise to nonlinear polynomial dependence in the effective theory. For example, in addition to terms linear in A_μ, the effective theory contains anomalous terms proportional to $F^{\mu\nu}\widetilde{F}_{\mu\nu}$ for $\pi^0 \to \gamma\gamma$ and $F^{\mu\nu}F_{\mu\nu}$ for $\sigma \to \gamma\gamma$, as well as non-anomalous powers of A_μ permitted by electromagnetic gauge invariance. Why should the rule $A_\mu \sim p$ for linear terms in the effective theory also hold for terms nonlinear in A_μ?

The reason is that infrared powers in NG-meson loop integrals are generated by a single mass scale: M. Therefore chiral order can be inferred from ordinary (non-operator) dimensionality. From $A_\mu \sim (\mathrm{length})^{-1}$, we can conclude e.g.

$$F^{\mu\nu}F_{\mu\nu} = O(p^4)\,. \tag{B4}$$

The extra two derivatives in F^2 compared with $A_\mu A_\nu$ are responsible for the failure of the σ vertex (72) to dominate one-loop meson contributions to $\sigma \to \gamma\gamma$.

When the lowest chiral order for a process induced by external currents mixes tree and loop diagrams, the rules of Appendix A must be amended. First, diagrams formed entirely from current vertices, i.e. with no vertices of the relevant χPT Lagrangian for strong interactions, should be classified according to their loop number: tree, one-loop, and sometimes higher. Then for each class, adding an internal loop produced by strong-interaction vertices increases the chiral order by at least 2. So the mixing of loop numbers for a given chiral order persists in higher orders, but the overall convergence rule that each new internal loop is suppressed by $M^2 \ln M$ or M^2 is maintained.

Evidently, gauge invariance and the restriction to currents as external operators is not essential for this discussion. All we need is a source $\mathcal{J}$ of unique (non-operator) dimensionality for a QCD Wilson operator. This will generate a polynomial in $\mathcal{J}$ for the effective theory with a chiral-order rule for $\mathcal{J}$. For example, let $\mathcal{S}$ and $\mathcal{P}$ be sources for $\bar{q}q$ and $\bar{q}\gamma_5 q$ in QCD corresponding to

$$\big\{U \pm U^\dagger\big\} e^{(3-\gamma_m)\sigma} \tag{B5}$$

in lowest-order $\chi\mathrm{PT}_\sigma$. Then insertion of this operator into a ϕ or σ propagator yields

$$O\big(1/p^2\big) \longrightarrow O\big(\{\mathcal{S} \text{ or } \mathcal{P}\}/p^4\big) \tag{B6}$$

so keeping the chiral order constant, we rediscover the rule [34]

$$\mathcal{S} \sim \mathcal{P} \sim O\big(p^2\big)\,. \tag{B7}$$

For a Wilson operator represented by a lowest order χPT operator whose vertices are of chiral order k, the result is

$$\mathcal{J} \sim O\big(p^{2-k}\big)\,. \tag{B8}$$

Appendix C: Electromagnetic Trace Anomaly in QCD

Originally, before QCD was invented, the electromagnetic trace anomaly was derived [37, 38] assuming a theory of broken scale invariance for strong interactions [81]:

$$\dim\, \theta^\mu_\mu < 4\,. \tag{C1}$$

This anomaly relates the amplitude $T\langle\theta^{\mu}_{\mu}J_{\alpha}J_{\beta}\rangle_{\rm vac}$ in the zero-energy limit (78) to the Drell-Yan ratio for the three-flavor theory at infinite[21] center-of-mass energy. Our application (79) is restricted to α_s being exactly at the fixed point $\alpha_{\rm IR}$, where broken scale invariance is still valid.

In the physical region $0 < \alpha_s < \alpha_{\rm IR}$ of QCD, broken scale invariance, with its anomalous power laws in the ultraviolet limit for all operators except conserved currents, is not valid, because the gluonic trace anomaly in Eq. (1) violates Eq. (C1). However the ultraviolet requirements of the derivation can be checked directly by using asymptotic freedom: the leading short-distance behaviors of

$$T\langle J_{\alpha}J_{\beta}\theta_{\mu\nu}\rangle_{\rm vac}\ ,\ T\langle J_{\alpha}J_{\beta}\theta_{\mu\nu}\rangle_{\rm vac}\ \ \text{and}\ \ T\langle J_{\alpha}J_{\beta}\theta^{\mu}_{\mu}\rangle_{\rm vac} \tag{C2}$$

are given by one-loop amplitudes with massless propagators, which is a special case of what was assumed for broken scale invariance. (The last amplitude in (C2) is needed to ensure convergence of Eq. (78) at $x \sim y \sim 0$.) The fact that some nonleading terms die off as inverse logarithms instead of inverse powers has no effect on the derivation.

So we conclude that the derivation can also be carried through for QCD in the physical region, despite the hard breaking of scale invariance by the gluonic term in θ^{μ}_{μ}. The result is an exact relation

$$F(0) = \frac{2R_{\rm UV}\alpha}{3\pi}\ \ \text{for}\ \ 0 < \alpha_s < \alpha_{\rm IR} \tag{C3}$$

in terms of the perturbative ratio $R_{\rm UV} = 2$ of Eq. (73).

Comparing Eqs. (79) and (C3), we see that $F(0)$ is discontinuous in α_s at $\alpha_{\rm IR}$. That is not a problem because the σ pole and charged $\pi^{\pm}, K^{\pm}$ loops can produce singular behaviour such as

$$\sim \frac{q^2}{m_{\sigma}^2 - q^2}\ \ \text{for}\ q, m_{\sigma} \sim 0\,. \tag{C4}$$

However a relation between $R_{\rm UV}$ and $R_{\rm IR}$ *cannot* be established because the condition (45) for $\mathcal{L}_{\rm anom}$ is not valid for an expansion not about $\alpha_{\rm IR}$.

[21] Note that the result (79) is exact; it is *not* due to an estimate at some large but finite energy. For example, it does not assume duality [120].

[1] I. Caprini, G. Colangelo, and H. Leutwyler, Phys. Rev. Lett. **96**, 132001 (2006).

[2] R. García-Martín, R. Kamiński, J. R. Peláez, and J. R. de Elvira, Phys. Rev. Lett. **107**, 072001 (2011).

[3] J. Beringer et al. (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).

[4] This article first appeared as a letter: R. J. Crewther and L. C. Tunstall, arXiv:1203.1321 [hep-ph]. For a brief introduction, see: R. J. Crewther and L. C. Tunstall, Mod. Phys. Lett. A **28**, 1360010 (2013).

[5] J. Ellis, Nucl. Phys. **B22**, 478 (1970).

[6] R. J. Crewther, Phys. Lett. B **33**, 305 (1970).

[7] S. L. Adler, J. C. Collins and A. Duncan, Phys. Rev. D **15**, 1712 (1977); P. Minkowski, Berne PRINT-76-0813, September 1976; N. K. Nielsen, Nucl. Phys. **B120**, 212 (1977); J. C. Collins, A. Duncan and S. D. Joglekar, Phys. Rev. D **16**, 438 (1977).

[8] J. Schechter, Phys. Rev. D **21**, 3393 (1980); A. A. Migdal and M. A. Shifman, Phys. Lett. B **114**, 445 (1982); J. Ellis and J. Lánik, Phys. Lett. B **150**, 289 (1985); J. F. Donoghue and H. Leutwyler, Z. Phys. C **52**, 343 (1991); G. E. Brown and M. Rho, Phys. Rev. Lett. **66**, 2720 (1991).

[9] B. Holdom, Phys. Rev. D **24**, 1441 (1981); Phys. Lett. B **150**, 301 (1985).

[10] T. Akiba and Y. Yanagida, Phys. Lett. B **169**, 432 (1986).

[11] T. Appelquist, D. Karabali and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D **35**, 774 (1987); **36**, 568 (1987).

[12] K. Yamawaki, M. Bando and K.I. Matumoto, Phys. Rev. Lett. **56**, 1335 (1986).

[13] William E. Caswell, Phys. Rev. Lett. **33**, 244 (1974); T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).

[14] T. Appelquist, G. T. Fleming and E. T. Neil, Phys. Rev. Lett. **100**, 171607 (2008); **102**, 149902(E) (2009); Phys. Rev. D **79**, 076010 (2009).

[15] L. Del Debbio, Proc. Sci. LATTICE2010 (2010) 004.

[16] M. Bando, K. Matumoto and K. Yamawaki, Phys. Lett. B **178**, 308 (1986).

[17] T. Appelquist and Y. Bai, Phys. Rev. D **82**, 071701(R) (2010).

[18] B. Holdom and J. Terning, Phys. Lett. B **187**, 357 (1987); **200**, 338 (1988).

[19] For a dissenting view, see: P. Minkowski, *Proc. Conf. M. Gell-Mann's* 80^{th} *birthday*, Singapore 2010, ed. H. Fritzsch and K. K. Phua (World Scientific 2011) p. 74; `http://www.mink.itp.unibe.ch/lectures.html`.

[20] G. Grunberg, Phys. Rev. D **29**, 2315 (1984).

[21] John M. Cornwall, Phys. Rev. D **26**, 1453 (1982); J. M. Cornwall and J. Papavassiliou, Phys. Rev. D **40**, 3474 (1989); A. C. Aguilar, D. Binosi and J. Papavassiliou, J. High Energy Phys. 07 (2010) 002.

[22] A. C. Mattingly and P. M. Stevenson, Phys. Rev. Lett. **69**, 1320 (1992); Phys. Rev. D **49**, 437 (1994); D. M. Howe and C. J. Maxwell, Phys. Lett. B **541**, 129 (2002); Phys. Rev. D **70**, 014002 (2004); P. M. Stevenson, Nucl. Phys. **B868**, 38 (2013); **B875**, 63 (2013).

[23] D. V. Shirkov and I. L. Solovtsov, Phys. Rev. Lett. **79**, 1209 (1997); J. A. Gracey, J. High Energy Phys. 05 (2006) 052; 02 (2010) 78(E).

[24] S. J. Brodsky, S. Menke, C. Merino and J. Rathsman, Phys. Rev. D **67**, 055008 (2003); S. J. Brodsky, G. F. de Teramond and A. Deur, Phys. Rev. D **81**, 096010 (2010).

[25] R. Horsley, H. Perlt, P. E. L. Rakow, G. Schierholz and A. Schiller, Phys. Lett. B **728**, 1 (2014).

[26] L. von Smekal, A. Hauck and R. Alkofer, Phys. Rev. Lett. **79**, 3591 (1997); C. S. Fischer and R. Alkofer, Phys. Rev. D **67**, 094020 (2003); H. Gies, Phys. Rev. D **66**, 025006 (2002); A. C. Aguilar, D. Binosi, J. Papavassiliou and J. Rodriguez-Quintero, Phys. Rev. D **80**, 085018 (2009).

[27] M. Lüscher, R. Sommer, P. Weisz, and U. Wolff, Nucl. Phys. **B413**, 481 (1994).

[28] S. Weinberg, Physica (Amsterdam) **96A**, 327 (1979).

[29] R. P. Feynman, in *Symmetries in Elementary Particle Physics*, ed. A. Zichichi (Academic, New York, 1965), p. 111.

[30] R. P. Feynman, M. Kislinger and F. Ravndal, Phys. Rev. D **3**, 2706 (1971).

[31] M. K. Gaillard and B. W. Lee, Phys. Rev. Lett. **33**, 108 (1974).

[32] G. Altarelli and L. Maiani, Phys. Lett. B **52**, 351 (1974).

[33] G. D'Ambrosio and D. Espriu, Phys. Lett. B **175**, 237 (1986); J. L. Goity, Z. Phys. C **34**, 341 (1987).

[34] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).

[35] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

[36] H. Leutwyler, Ann. Phys. (N.Y.) **235**, 165 (1994).

[37] R. J. Crewther, Phys. Rev. Lett. **28**, 1421 (1972).

[38] M. S. Chanowitz and J. Ellis, Phys. Lett. B **40**, 397 (1972); Phys. Rev. D **7**, 2490 (1973).

[39] P. Carruthers, Phys. Rep. C **1**, 1 (1971).

[40] L.-F. Li and H. Pagels, Phys. Rev. Lett. **26**, 1204 (1971); **27**, 1089 (1971); Phys. Rev. D **5**, 1509 (1972).

[41] H. Pagels, Phys. Rep. C **16**, 219 (1975).

[42] E. Golowich, Phys. Rev. D **23**, 2610 (1981).

[43] M. K. Volkov, A. N. Ivanov and N. I. Troitskaya, Yad. Fiz. **47**, 1157 (1988) [trans. Sov. J. Nucl. Phys. **47** 736 (1988)].

[44] T. Morozumi, C. S. Lim, and A. I. Sanda, Phys. Rev. Lett. **65**, 404 (1990).

[45] A. D. Polosa, N. A. Törnqvist, M. D. Scadron, and V. Elias, Mod. Phys. Lett. A **17**, 569 (2002); *Workshop on Scalar Mesons and Related Topics (Scadron70)*, ed. G. Rupp *et al.*, AIP Conf. Proc. **1030** (AIP 2008).

[46] T. N. Truong, Acta Phys. Pol. B **15**, 633 (1984); A. Dobado, M. J. Herrero and T. N. Truong, Phys. Lett. B **235**, 134 (1990).

[47] T. N. Truong, Phys. Lett. B **99**, 154 (1981).

[48] A. Neveu and J. Scherk, Ann. Phys. (N.Y.) **57**, 39 (1970).

[49] T. N. Truong, Phys. Lett. B **207**, 495 (1988).

[50] C. Roiesnel and T. N. Truong, Nucl. Phys. **B187**, 293 (1981).

[51] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 539 (1985); A.V. Anisovich and H. Leutwyler, Phys. Lett. B **375**, 335 (1996).

[52] U.-G. Meissner, Comments Nucl. Part. Phys. **20**, 119

(1991); Rep. Prog. Phys. **56**, 903 (1993).
[53] G. Ecker, A. Pich, and E. de Rafael, Phys. Lett. B **189**, 363 (1987).
[54] J. Kambor and B. R. Holstein, Phys. Rev. D **49**, 2346 (1994).
[55] J. Bijnens and F. Cornet, Nucl. Phys. **B296**, 557 (1988).
[56] J. F. Donoghue, B. R. Holstein, and Y. C. Lin, Phys. Rev. D **37**, 2423 (1988).
[57] J. F. Donoghue and B. R. Holstein, Phys. Rev. D **48**, 137 (1993).
[58] H. Marsiske *et al.* (Crystal Ball Collaboration), Phys. Rev. D **41**, 3324 (1990).
[59] Particle Data Group, Phys. Lett. B **50**, 1 (1974).
[60] N. A. Törnqvist and M. Roos, Phys. Rev. Lett. **76**, 1575 (1996).
[61] J. F. Donoghue. in *Chiral Dynamics of Hadrons and Nuclei*, ed. D. P. Min and M. Rho (Seoul National University Press, Seoul, 1995), p. 87, hep-ph/9506205.
[62] G. Colangelo, S. Lanz, H. Leutwyler and E. Passemar, Proc. Sci. EPS-HEP2011 (2011) 304.
[63] M. R. Pennington, Phys. Rev. Lett. **97**, 011601 (2006); Mod. Phys. Lett. A **22**, 1439 (2007).
[64] R. García-Martín, J. R. Peláez and F. J. Ynduráin, Phys. Rev. D **76**, 074034 (2007).
[65] G. Colangelo, E. Passemar and P. Stoffer, EPJ Web Conf. **37**, 05006 (2012).
[66] N. N. Trofimenkoff, arXiv:1202.6254v1 [hep-ph].
[67] S. Weinberg, Phys. Rev. Lett. **16**, 879 (1966); **17**, 616 (1966); **18**, 188 (1967); **18**, 507 (1967); Phys. Rev. **166**, 1568 (1968).
[68] R. Dashen and M. Weinstein, Phys. Rev. **183**, 1261 (1969).
[69] J. Gasser and H. Leutwyler, Phys. Lett. B **125**, 321 (1983); **125**, 325 (1983).
[70] M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. **175**, 2195 (1968).
[71] J. Gasser and H. Leutwyler, Phys. Rep. C **87**, 77 (1982).
[72] S. Scherer and M. R. Schindler, Lect. Notes Phys. **830**, 1 (2012).
[73] J. Gasser, C. Haefeli, M. A. Ivanov and M. Schmid, Phys. Lett. B **652**, 21 (2007); **675**, 49 (2009).
[74] J. Nebreda and J. R. Peláez, Phys. Rev. D **81**, 054035 (2010).
[75] A. Manohar and H. Georgi, Nucl. Phys. **B234**, 189 (1984); H. Georgi, Phys. Lett. B **298**, 187 (1993).
[76] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[77] G. Veneziano, Nucl. Phys. **B117**, 519 (1976).
[78] E. Witten, Nucl. Phys. **B160**, 57 (1979).
[79] R. L. Jaffe, Phys. Rev. D **15**, 267 (1977); **15**, 281 (1977).
[80] J. R. Peláez, M. R. Pennington, J. Ruiz de Elvira and D. J. Wilson, Phys. Rev. D **84**, 096006 (2011); J. Nieves, A. Pich and E. Ruiz Arriola, Phys. Rev. D **84**, 096002 (2011).
[81] K. G. Wilson, Phys. Rev. **179**, 1499 (1969).
[82] R. S. Chivukula, A. G. Cohen, H. Georgi, B. Grinstein and A. V. Manohar, Ann. Phys. (N.Y.) **192**, 93 (1989).
[83] J. Ellis, in *Broken Scale Invariance and the Light Cone, 1971 Coral Gables Conference on Fundamental Interactions at High Energy*, Vol. 2, ed. M. Dal Cin *et al.* (Gordon and Breach, New York 1971), p. 77.
[84] C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) **59**, 42 (1970).
[85] A. Salam and J. Strathdee, Phys. Rev. **184**, 1760 (1969); C. J. Isham, A. Salam, and J. Strathdee, Phys. Lett. B **31**, 300 (1970); C. J. Isham, A. Salam, and J. Strathdee, Phys. Rev. D **2**, 685 (1970).
[86] J. S. R. Chisholm, Nucl. Phys. **26**, 469 (1961).
[87] S. Kamefuchi, L. O'Raifeartaigh and A. Salam, Nucl. Phys. **28**, 529 (1961).
[88] H. Kleinert and P. H. Weisz, Nucl. Phys. **B27**, 23 (1971); Nuovo Cimento A **3**, 479 (1971).
[89] A. Calle Cordón and E. Ruiz Arriola, AIP Conf. Proc. **1030**, 334 (2008); Phys. Rev. C **81**, 044002 (2010).
[90] R. G. Stuart, Phys. Lett. B **262**, 113 (1991); **272**, 353 (1991); A. Sirlin, Phys. Rev. Lett. **67**, 2127 (1991); Phys. Lett. B **267**, 240 (1991); F. Jegerlehner, M. Yu. Kalmykov and O. Veretin, Nucl. Phys. **B641**, 285 (2002); **B658**, 49 (2003).
[91] D. Djukanovic, J. Gegelia, A. Keller, and S. Scherer, Phys. Lett. B **680**, 235 (2009).
[92] G. Ecker, J. Gasser, A. Pich, and E. de Rafael, Nucl. Phys. **B321**, 311 (1989); J. F. Donoghue, C. Ramirez, and G. Valencia, Phys. Rev. D **39**, 1947, 1989.
[93] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[94] J. A. Oller and L. Roca, Eur. Phys. J. A **37**, 15 (2008).
[95] Y. Mao, X.-G. Wang, O. Zhang, H. Q. Zheng and Z. Y. Zhou, Phys. Rev. D **79**, 116008 (2009).
[96] B. Moussallam, Eur. Phys. J. C **71**, 1814 (2011).
[97] M. Hoferichter, D. R. Phillips and C. Schat, Eur. Phys. J. C **71**, 1743 (2011).
[98] R. Garcia-Martin and B. Moussallam, Eur. Phys. J. C **70**, 155 (2010).
[99] J. Gasser, M. A. Ivanov and M. E. Sainio, Nucl. Phys. **B728**, 31 (2005).
[100] J. Gasser, M. A. Ivanov and M. E. Sainio, Nucl. Phys. **B745**, 84 (2006).
[101] S. Bellucci, J. Gasser and M. E. Sainio, Nucl. Phys. **B423**, 80 (1994); **B431**, 413(E) (1994).
[102] L. Ametller, J. Bijnens, A. Bramon and F. Cornet, Phys. Lett. B **276**, 185 (1992).
[103] N. N. Achasov and G. N. Shestakov, Phys. Rev. D **77**, 074020 (2008).
[104] G. Mennessier, S. Narison and W. Ochs, Phys. Lett. B **665**, 205 (2008).
[105] M. Gell-Mann and A. Pais, in *Proceedings of the 1954 Glashow Conference on Nuclear and Meson Physics*, ed. E. H. Bellamy and R. G. Moorhouse (London, New York, Pergamon Press, 1955); M. Gell-Mann, in *Pions to Quarks: Particle Physics in the 1950s*, ed. L. M. Brown, M. Dresden, and L. Hoddeson (Cambridge University Press, Cambridge and New York 1989), p. 694, reprinted in *Murray Gell-Mann: selected papers.* ed. H. Fritzsch (World Scientific, Singapore, 2010), p. 360.
[106] J. Kambor, J. H. Missimer and D. Wyler, Nucl. Phys. **B346**, 17 (1990); Phys. Lett. B **261**, 496 (1991); G. Ecker, J. Kambor and D. Wyler, Nucl. Phys. **B394**, 101 (1993); J. M. Gerard and J. Weyers, Phys. Lett. B **503**, 99 (2001).
[107] J. A. Cronin, Phys. Rev. **161**, 1483 (1967).
[108] R. J. Crewther, Nucl. Phys. **B264**, 277 (1986).
[109] C. Bernard, T. Draper, A. Soni, H. D. Politzer, and M. B. Wise, Phys. Rev. D **32**, 2343 (1985).
[110] R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (John Wiley & Sons, Inc., New York, 1969), p. 545.
[111] A. I. Vainshtein, V. I. Zakharov and M. A. Shifman, JETP Lett. **22**, 55 (1975) [Pisma Zh. Eksp. Teor. Fiz.

22, 123 (1975)]; Nucl. Phys. **B120**, 316 (1977).

[112] M. Fukugita, T. Inami, N. Sakai and S. Yazaki, Phys. Lett. B **72**, 237 (1977); D. Tadić and J. Trampetić, Phys. Lett. B **114**, 179 (1982); A. J. Buras and J.-M. Gérard, Nucl. Phys. **B264**, 371 (1986); R. S. Chivukula, J. M. Flynn and H. Georgi, Phys. Lett. B **171**, 453 (1986); W. A. Bardeen, A. J. Buras and J.-M. Gérard, Phys. Lett. B **180**, 133 (1986); Nucl. Phys. **B293**, 787 (1987); Phys. Lett. B **192**, 138 (1987); J. Bijnens and B. Guberina, Phys. Lett. B **205**, 103 (1988).

[113] A. J. Buras, J.-M. Gérard, and W. A. Bardeen, Eur. Phys. J. C **74**, 2871 (2014).

[114] B. Guberina, A. Pich and E. de Rafael, Phys. Lett. B **163**, 198 (1985); A. Pich, B. Guberina and E. de Rafael, Nucl. Phys. **B277**, 197 (1986); A. Pich and E. de Rafael, Phys. Lett. B **189**, 369 (1987); Nucl. Phys. **B358**, 311 (1991).

[115] V. A. Miransky, Nuovo Cim. A **90**, 149 (1985).

[116] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[117] J. C. Pati and C. H. Woo, Phys. Rev. D **3**, 2920 (1971); K. Miura and T. Minamikawa, Prog. Theor. Phys. **38**, 954 (1967).

[118] S. Matsuzaki and K. Yamawaki, Phys. Rev. Lett. **113**, 082002 (2014).

[119] N. V. Krasnikov, Nucl. Phys. **B192**, 497 (1981).

[120] E. Etim and M. Greco, Lett. Nuovo Cimento **12**, 91 (1975); M. Shifman, Czech. J. Phys. B **52**, 102 (2002).